"Never imagine yourself not to be otherwise than what it might appear to others that what you were or might have been was not otherwise than what you had been would have appeared to them to be otherwise."

Lewis Carroll

Alice in Wonderland

THE UNIVERSITY OF ALBERTA

# DIFFERENCE FREQUENCY HARMONIC ION HEATING VIA HIGH FREQUENCY INCIDENT FIELDS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Difference Frequency Harmonic Ion Heating Via High Frequency Incident Fields" submitted by Clement David Fournier in partial fulfilment of the requirements for the degree of Master of Science.

# ABSTRACT

The heating of ions in a magnetized plasma by the use of the second order fields generated by the nonlinear mixing of two high frequency waves ( high frequency means $\omega > \omega_{p-}$ ). The kinetic equations describing the mixing and heating process are solved using the method of orbit integrations. The energy is absorbed from the second order fields via cyclotron damping. The sensitivity of the absorbed power to fluctuations in frequency, density, static magnetic field and mixing angle is analysed.

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Table of Contents

## List of Symbols

| | |
|---|---|
| $\delta_{ei}$ | Electron-ion Collision frequency |
| $\delta_{ee}$ | Electron-electron collision frequency |
| $\omega_{p-}$ | Electron plasma frequency |
| $\omega_{p+}$ | Ion plasma frequency |
| $\Omega_+$ | Ion gyration frequency |
| $\Omega_-$ | Electron gyration frequency |
| $\underline{E}$ | Electric field intensity, statvolts/cm. |
| $\underline{B}$ | Magnetic field in gauss |
| $T$ | Temperature, $^o$k |
| $n_+$ | Ion number density |
| $n_-$ | Electron number density |
| $e$ | Electron charge |
| $K$ | Boltzmann's constant |
| $c$ | Propagation velocity of light |
| $\omega$ | Frequency of difference frequency wave |
| $\omega_1, \omega_2$ | Frequencies of incident electromagnetic waves |
| $\underline{k}$ | Propagation vector $\lvert \underline{k} \rvert = \omega / c$ |
| $\underline{v}$ | Particle velocity |
| $\hat{x}, \hat{y}, \hat{z}$ | Unit Vectors: Refers to the 3 cartesian co-ordinates |
| $\epsilon$ | Refers to the sign of the charge |
| $\perp$ & $\parallel$ | Refers to perpendicular and parallel to the static magnetic field |
| $\langle\ \rangle_\perp$ | Average over $V_x$ and $V_y$ |
| $\langle\ \rangle$ | Average over $V_x$, $V_y$ and $V_z$ |

A vector, $\mathbf{V}$ , is denoted $\underset{\sim}{V}$

A tensor, $\mathbf{M}$ , is denoted $\underset{\approx}{M}$

$\underset{\sim}{B}^*$ Complex conjugate of the vector $\underset{\sim}{B}$

$\mathcal{R}e$ Real part of

$\mathcal{I}m$ Imaginary part of

Table of Integrals

$$F_0(s) = \pi^{1/2} \frac{K_z}{|K_z|} e^{-s^2} + 2i e^{-s^2} \int_0^s e^{t^2} dt$$

$$F_1(s) = -i \left( \frac{2KT}{m} \right)^{1/2} + \left( \frac{\omega + n\Omega}{K_z} \right) F_0(s)$$

$$F_2(s) = -\frac{i(\omega+n\Omega)}{K_z} \left( \frac{2KT}{m} \right)^{1/2} + \left( \frac{\omega+n\Omega}{K_z} \right)^2 F_0(s)$$

$$F_3(s) = -\frac{i}{2} \left( \frac{2KT}{m} \right)^{3/2} - i \left( \frac{\omega+n\Omega}{K_z} \right)^2 \left( \frac{2KT}{m} \right)^{1/2} + \left( \frac{\omega+n\Omega}{K_z} \right)^3 F_0(s)$$

# CHAPTER 1 INTRODUCTION

Controlled thermonuclear fusion has been on the minds of engineers and physicists since the explosion of the first hydrogen bomb. A large amount of time and money has been spent in the last two decades to achieve this end. Although considerable progress has been made towards achieving this goal, the basic problems still to be surmounted are essentially the same as they were at the onset of research. The two basic problems associated with fusion are confinement and heating.

Since this thesis deals only with plasma heating, a discussion of the state of the research in plasma confinement will not be given. (For a detailed discussion of plasma confinement techniques, see references 1-4.)

To achieve positive energy yields from a fusion reaction, it is estimated[5] that ion temperatures of $10^8$ °K or better will be needed. A first-order calculation of electron-electron and electron-ion collision frequencies done by Spitzer[6] gives

$$\delta_{ee} = \left(\frac{2\pi}{m_i}\right)^{1/2} \frac{n_e e^4}{(KT_e)^{3/2}} \ln \Lambda \qquad (1.1)$$

$$\text{\&} \quad \Lambda = \left(\frac{KT_e}{e}\right)^{3/2} \frac{1}{\sqrt{4\pi n_0}}$$

and

$$\delta_{ei} = \left(\frac{2\pi}{m_e}\right)^{1/2} \frac{n_i}{(kT)^{3/2}} \ln \Lambda \qquad (1.2)$$

Equations 1.1 and 1.2 show that the collision frequencies in a plasma are inversely proportional to $T^{3/2}$. This in turn implies that any heating method which depends on electron-electron collisions for energy absorption will become more and more inefficient as the temperature of the electrons increases. Equation 1.2 indicates that ion heating via the equipartition of energy from the electrons is a poor mechanism at high temperatures. From the conservation of momentum, $n_i \delta_{ie} m_i = n_e \delta_{ei} m_e$. For the case where $n_i = n_e$ (required for charge neutrality in a two component plasma).

$$\delta_{ie} = \delta_{ei} \frac{m_e}{m_i} \qquad (1.3)$$

Equation 1.3 implies that if the ions are selectively heated (ion-electron equipartition times are $m_i/m_e$ larger than electron-ion equipartition times) a significant amount of electron-ion temperature decoupling can occur. This means that the ions can in principle be maintained at a higher temperature than the electrons. This property of a plasma is very important since in order to achieve fusion one should "ideally" have ions at $10^8$ °K and cold electrons. This is because the electrons, due to their high mobility, are responsible for the majority of the losses from a hot plasma. The two most important mechanisms for losses via electrons are cyclotron and bremsstrahlung radiation. It is clear from this argument

that the most efficient heating methods to be employed for fusion will be those which selectively heat the ions through a non-collisional mechanism. In order to achieve this end a considerable amount of research in the field of plasma heating has been directed at the problem of ion energy absorption via cyclotron damping.

Heating Techniques Proposed to Date

In this section a brief review of the various methods used to date for plasma heating is given.

1. Ohmic Heating

Plasma Ohmic heating using high intensity laser radiation has recently been proposed as a method by which a plasma can be brought to thermonuclear temperatures. Ohmic heating has been used in fusion machines for preheating the plasma.[7] Ohmic heating arises strictly from the resistivity of the plasma to the electric field. Since the collision frequency varies as $T^{-3/2}$, this method of heating becomes more and more inefficient as the electron temperature increases. Another disadvantage of this method is that most of the power is absorbed by electrons. At high temperatures, the electron-ion equipartition time becomes large, making the transfer of energy between the electrons and ions very inefficient.

2. Magnetic Pumping

In this scheme,[8,9,10] the magnetic field confining

the plasma is modulated in time. Heating in the perpendicular direction occurs because of the adiabaticity of the magnetic moment and the absorbed energy is transferred via collisions to the parallel direction.[8,9] This technique involves the ion-ion relaxation time and is therefore very inefficient at high temperatures. It also suffers from the fact that at high frequencies (where the heating would presumably be more efficient) only a very thin surface layer of the plasma is compressed and only the surface layer is being heated.

3. Transit-Time Heating

In transit-time heating, the magnetic field is modulated in time and space. In this method,[8] the longitudinal adiabatic invariant is responsible for energy absorption and the randomization occurs via large scale scattering. A serious drawback to this method is the splitting up of the original Maxwellian distribution into a fast and slow component leading to instabilities in velocity space. Numerous theoretical and experimental investigations carried out in the last few years on transit-time heating have shown this scheme to be the strongest contender for heating plasmas in toroidal geometries.[9] A more detailed discussion of magnetic pumping can be found in James and Capjack.[11]

4. Rapid Compression

A preheated plasma column may be heated through a radial compression obtained by pulsing up the strength of the magnetic field. The velocity imparted to the particles using rapid compression is species independent. This species independence implies that the ions are gaining most of the energy from the compression. An experiment utilizing this type of heating scheme is the 2X mirror experiment at the Lawrence Radiation Laboratory.[12]

5. Ion Cyclotron Resonance Heating (I.C.R.H.)

Plasma heating by the use of frequencies at or near the ion cyclotron frequency has been under intense theoretical and experimental investigation.[13-17]

a. Reverse Turn Induction Coils

The coil is often referred to as a Stix coil and is used to generate cyclotron waves in a plasma loaded cylindrical wave guide.[18] The Stix coil configuration has been used in the B-65 stellerator at Princeton and has also been used in the model C-stellerator.[19] The coil is built so that current passing through a section is $180^{o}$ out of phase with the current in the adjacent sections. This allows the electrons to flow along the magnetic field lines to cancel the radial oscillating space charge electric field that would otherwise be set-up, resulting in the plasma shielding itself from the applied field.

b. Coaxial Electrodes

R.F. power is applied to coaxial electrodes of a mirror device and a torsional Alfven wave is generated in the plasma. Since the electric field of the wave in the plasma is in the radial direction, an efficient coupling of energy is possible. The theoretical coupling efficiency can be estimated by treating the device as a coaxial wave guide filled with a material of dielectric constant $K = 1 + (4\pi\rho c^2)/B^2$ The waves are generated in a region where $\omega/\Omega_+ \simeq .5$ Absorption of the wave energy is then accomplished through the use of a magnetic beach.

In an experiment performed by Boley, Wilcox, et al,[20] the transfer of energy from a 1MW, 8.3 MHz oscillator to a plasma with a density of $6 \times 10^{12}$ $cm^{-3}$ was at an efficiency of 65%. Better than 90% of the wave energy was absorbed by the ions at the magnetic beach.

A similar experiment was performed by Shvets et al[21,22] on the Vikhr (Whirlwind) device. In this experiment an ion temperature of 250 e.v. was obtained in a plasma with a density $2 \times 10^{14}$ $cm^{-3}$.

c. Difference Frequency Ion Heating

Since the topic of this thesis is plasma heating via non-linear mixing of waves in a plasma, a more detailed discussion of the history of this method will be given.

A large amount of theoretical and experimental work has been done on non-linear mixing of waves in plasmas.[23-35] In 1966, James and Thompson[36] proposed using the difference frequency harmonic which is generated by the non-linear mixing of two high frequency waves for plasma heating. (high frequency waves are defined as $\omega_1, \omega_2 > \omega_{p-}$ to ensure penetration of the incident waves into the bulk of the plasma.) In order to enhance the second-order fields, they tuned their difference frequency to a natural resonance in the plasma. Using a plasma natural resonance made the absorbed power critically dependent on plasma parameters such as number density and magnetic field strength. Stability of plasma parameters of better than one part in $10^5$ were required in order to maintain the absorbed power at an acceptable level. They also found that the power absorbed from the difference frequency harmonic varied as $|E_1|^4/\omega_1^4$. This meant that for $\omega_1 > \omega_{p-}$ the power absorbed would be very small unless the electric fields were very large. In order to lower the incident frequencies and still ensure good penetration of the incident waves in the plasma, an analysis was done by Capjack and James[37] using incident fields in the whistler regime. In this analysis, the difference frequency wave was tuned to the ion cyclotron resonance and a full kinetic analysis was done to calculate the power absorbed by cyclotron damping. This analysis also proved to have very high requirements on the static fields and the difference frequency.

In order to circumvent these problems, a kinetic analysis of cyclotron damping of high frequency waves is undertaken in this thesis. The incident frequencies used in the numerical calculations are chosen such that a $CO_2$ laser can be used to experimentally verify the results of this analysis. The difference frequency wave is set equal to the ion cyclotron frequency. The power absorbed by the ions and electrons at the difference frequency is computed. Cyclotron energy absorption proves to be relatively insensitive to magnetic field, difference frequency and number density fluctuations. This method of energy absorption shows an improvement as the number density of the plasma increases. These characteristics make this method a good candidate for heating to thermonuclear temperatures.

# CHAPTER 2 NONLINEAR INTERACTION OF WAVES IN A PLASMA

In this chapter the collisionless Boltzmann equation will be solved to second order to describe the nonlinear mixing of two high frequency waves in a plasma. To solve the Boltzmann equation the method of orbit integration, first used by Drummond[38] to calculate first order fields and currents and latter extended by Capjack and James[37] to include second order effects, will be used.

## 2.1 Kinetic Equation

Consider a particle trajectory given by

$$\underset{\sim}{r} = \underset{\sim}{r}(t)$$

Then the total rate of change of the j'th distribution along this trajectory is given in the Lagrangian coordinates along the trajectory by

$$\frac{df_j}{dt} = \frac{\partial f_j}{\partial t} + \frac{d\underset{\sim}{r}}{dt}\cdot\frac{\partial f_j}{\partial \underset{\sim}{r}} + \frac{d\underset{\sim}{v}}{dt}\cdot\frac{\partial f_j}{\partial \underset{\sim}{v}} \tag{2.1}$$

Setting $\underset{\sim}{v} = \frac{d\underset{\sim}{r}}{dt}$, using the Lorentz force equation to replace $\frac{d\underset{\sim}{v}}{dt}$ by $\frac{Z_j e}{m_j}\left[\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right]$, and taking the limit of no collisions ( $\frac{df_j}{dt} = 0$ ) equation 2.1 becomes the Boltzmann-Vlasov equation.

$$\frac{\partial f_j}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial f_j}{\partial \underset{\sim}{r}} + \frac{Z_j e}{m_j}\left[\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right]\cdot\frac{\partial f_j}{\partial \underset{\sim}{v}} = 0 \tag{2.2}$$

The Vlasov equation coupled with Maxwell's equations for the E.M. field

$$\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t} \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} \qquad \nabla \cdot \underset{\sim}{E} = 4\pi\rho \tag{2.3}$$

and with the definition of charge and current

$$\rho = \sum_k e z_k \epsilon \iiint f_k \, d\underset{\sim}{v}$$

$$\underset{\sim}{j} = \sum_k e z_k \epsilon \iiint \underset{\sim}{v} f_k \, d\underset{\sim}{v}$$

form a closed set of equations which can, in principle, be solved for $f_j$ .

[note: The validity of using the Boltzmann-Vlasov equation is discussed in standard books of Plasma Physics (i.e. Gartenhaus-Elements of Plasma Physics-chapter 4) and will not be discussed here. It will also be implicitely assumed that a second order perturbation expansion of $f_j$ provides an "adequate" description of wave mixing in a hot magneto-plasma.]

## 2.2 Solution of the Kinetic Equation

The solution to the set of equations 2.2 and 2.3 is taken to be an expansion to second-order of the form:

$$\begin{bmatrix} f^{\pm} \\ \underset{\rightarrow}{E} \\ \underset{\rightarrow}{B} \end{bmatrix} = \begin{bmatrix} f_0^{\pm} \\ 0 \\ \underset{\rightarrow}{B}_0 \end{bmatrix} + \begin{bmatrix} f_1^{\pm} \\ \underset{\rightarrow}{E}_1 \\ \underset{\rightarrow}{B}_1 \end{bmatrix} e^{i(\underset{\rightarrow}{K}_1\cdot\underset{\rightarrow}{r} - \omega_1 t)} + \begin{bmatrix} f_2^{\pm} \\ \underset{\rightarrow}{E}_2 \\ \underset{\rightarrow}{B}_2 \end{bmatrix} e^{i(\underset{\rightarrow}{K}_2\cdot\underset{\rightarrow}{r} - \omega_2 t)}$$

$$+ \begin{bmatrix} f_3^{\pm} \\ \underset{\rightarrow}{E}_3 \\ \underset{\rightarrow}{B}_3 \end{bmatrix} e^{i(\underset{\rightarrow}{K}\cdot\underset{\rightarrow}{r} - \omega t)} + \text{complex conjugate terms.} \qquad (2.4)$$

where $\underset{\rightarrow}{K} = \underset{\rightarrow}{K}_1 - \underset{\rightarrow}{K}_2$ and $\omega = \omega_1 - \omega_2$

The $\pm$ signs refer to the sign of the charge of the two component plasma under consideration. The $f_0^{\pm}$, $f_{1,2}^{\pm}$, $f_3^{\pm}$ are the zeroeth, first and second order terms to the distribution function. To make the mathematical analysis of this problem tractable it will be assumed that

$$\begin{aligned} \underset{\rightarrow}{B}_0 &= B_0 \hat{z} \\ \underset{\rightarrow}{K}_1 &= K_{1x}\hat{x} + K_{1z}\hat{z} \\ \underset{\rightarrow}{K}_2 &= K_2 \hat{z} \\ \underset{\rightarrow}{E}_1 &= E_{1x}\hat{x} + E_{1z}\hat{z} \\ \underset{\rightarrow}{E}_2 &= E_2 \hat{x} \end{aligned} \qquad (2.5)$$

### 2.2.1 Solution of the Zeroeth-Order Equation

The zeroeth-order trajectory is defined by the equation of motion

$$\underset{\to}{v} = \frac{d\underset{\to}{r}}{dt} \qquad\qquad \frac{d\underset{\to}{v}}{dt} = \frac{Z e \epsilon}{m_\pm} \underset{\to}{v} \times \underset{\to}{B}_o \tag{2.6}$$

The rate of change of $f^\pm$ along the zeroeth order trajectory is given by

$$\left(\frac{df^\pm}{dt}\right)_o = \frac{\partial f^\pm}{\partial t} + \underset{\to}{v}\cdot\frac{\partial f^\pm}{\partial \underset{\to}{r}} + \frac{Ze\epsilon}{m_\pm}\underset{\to}{v}\times\underset{\to}{B}_o\cdot\frac{\partial f^\pm}{\partial \underset{\to}{v}} \tag{2.7}$$

The rate of change of the zeroeth-order solution of the B.V. equation along the zeroeth-order trajectory is

$$\left(\frac{df_o^\pm}{dt}\right)_o = \frac{\partial f_o^\pm}{\partial t} + \underset{\to}{v}\cdot\frac{\partial f_o^\pm}{\partial \underset{\to}{r}} + \frac{Ze\epsilon}{m_\pm}\underset{\to}{v}\times\underset{\to}{B}_o\cdot\frac{\partial f_o^\pm}{\partial \underset{\to}{v}} = 0 \tag{2.7A}$$

from the B.V. assumption. The time and space independent solution of 2.7A is the form $f_o^\pm = f_o^\pm(V_\perp^2, V_{\parallel})$ where $\perp$ & $\parallel$ refer to directions with respect to the static magnetic field. In this analysis it will be assumed that the zeroeth-order distribution is Maxwellian.

i.e.
$$f_o^\pm = \left(\frac{m^\pm}{2\pi KT_\pm}\right)^{3/2} e^{-\frac{mv^2}{2KT_\pm}} \tag{2.8}$$

### 2.2.2 First-Order Solution

The first order solution to the Boltzmann equation is obtained by expanding to first order:

$$f^\pm = f_o^\pm + f_1^\pm$$

Using this expansion in 2.2 and retaining terms of first order yields:

$$0 = \frac{\partial f_1^\pm}{\partial t} + \underset{\to}{v}\cdot\frac{\partial f_1^\pm}{\partial \underset{\to}{r}} + \frac{Ze\epsilon}{m_\pm}\left(\frac{\underset{\to}{v}}{c}\times B_o\right)\cdot\frac{\partial f_1^\pm}{\partial \underset{\to}{v}} + \frac{Ze\epsilon}{m_\pm}\left(\underset{\to}{E}_1 + \frac{\underset{\to}{v}}{c}\times\underset{\to}{B}_1\right)\cdot\frac{\partial f_o^\pm}{\partial \underset{\to}{v}} \tag{2.9}$$

If the total derivative of $f_1^{\pm}$ is taken along the zeroeth-order trajectory and the result substituted in 2.9, then 2.9 becomes:

$$\left(\frac{d f_1^{\pm}}{dt}\right)_0 = -\frac{ze\epsilon}{m_\pm}\left[\underset{\sim}{E}_1 + \frac{\underset{\sim}{v}\times\underset{\sim}{B}_1}{c}\right]\cdot\frac{\partial f_0^{\pm}}{\partial \underset{\sim}{v}}$$

This equation can be integrated along the zeroeth-order trajectory to yield:

$$f_1^{\pm} = -\frac{ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(\underset{\sim}{E}_1' + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_1'}{c}\right)\cdot\frac{\partial f_0^{\pm\prime}}{\partial \underset{\sim}{v}'}\,dt' \qquad (2.10)$$

[Note; Rigorously speaking, the integration should have been written:

$$f_1^{\pm} = -\frac{ze\epsilon}{m}\int_{t_0}^{t}\left(\underset{\sim}{E}_1' + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_1}{c}\right)\cdot\frac{\partial f_0^{\pm\prime}}{\partial \underset{\sim}{v}'}\,dt' + f_1^{\pm}(\underset{\sim}{r},\underset{\sim}{v},t_0)$$

However, if attention is focused only on solutions that are growing in time; then $f_1^{\pm}(\underset{\sim}{r},\underset{\sim}{v},t_0)$ becomes exponentially small as $t_0 \rightarrow -\infty$ and may therefore be neglected.]

## 2.3 Second-Order Distribution Function

A second-order solution to the B.V. equation is obtained by using the expansions 2.4 in equation 2.2.

$$0 = \frac{\partial}{\partial t}\left[f_0^{\pm} + f_1^{\pm}e^{i(\underset{\sim}{k}_1\cdot\underset{\sim}{r}-\omega_1 t)} + f_2^{\pm}e^{i(\underset{\sim}{k}_2\cdot\underset{\sim}{r}-\omega_2 t)} + f_3^{\pm}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} + c.c.\right]$$

$$+ \underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}}\left[f_0^{\pm} + f_1^{\pm}e^{i(\underset{\sim}{k}_1\cdot r-\omega_1 t)} + f_2^{\pm}e^{i(k_2\cdot r-\omega_2 t)} + f_3^{\pm}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right. \qquad (2.11)$$

$$\left.+ c.c.\right] + \frac{ze\epsilon}{m_\pm}\left[\frac{\underset{\sim}{v}}{c}\times\left(\underset{\sim}{B}_0 + \underset{\sim}{B}_1 e^{i(\underset{\sim}{k}_1\cdot r-\omega_1 t)} + \underset{\sim}{B}_2 e^{i(\underset{\sim}{k}_2\cdot r-\omega_2 t)} + B_3 e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)\right.$$

$$\left.+\left(E_1 e^{i(\underset{\sim}{k}_1\cdot r-\omega_1 t)} + E_2 e^{i(\underset{\sim}{k}_2\cdot r-\omega_2 t)} + E_3 e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right) + c.c.\right]\cdot\frac{\partial}{\partial \underset{\sim}{v}}\left[f_0^{\pm}\right.$$

$$\left.+ \underline{f_1^{\pm}}e^{i(\underset{\sim}{k}_1\cdot r-\omega_1 t)} + f_2^{\pm}e^{i(\underset{\sim}{k}_2\cdot\underset{\sim}{r}-\omega_2 t)} + f_3^{\pm}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} + c.c.\right]$$

Using harmonic balance to equate terms with common exponential $e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}$ where $\underset{\sim}{k}=\underset{\sim}{k}_1-\underset{\sim}{k}_2$ and $\omega=\omega_1-\omega_2$

yields:

$$\frac{\partial f_3^{\pm}}{\partial t}+\underset{\sim}{v}\cdot\frac{\partial f_3^{\pm}}{\partial \underset{\sim}{r}}+\frac{ze\epsilon}{m_\pm}\left(\frac{\underset{\sim}{v}}{c}\times \underset{\sim}{B}_0\right)\cdot\frac{\partial f_3^{\pm}}{\partial \underset{\sim}{v}}=-\frac{ze\epsilon}{m_\pm}\Big[\left(E_3+\frac{\underset{\sim}{v}\times B_3}{c}\right)\cdot\frac{\partial f_0^{\pm}}{\partial \underset{\sim}{v}} \tag{2.12}$$

$$\left(E_1+\frac{\underset{\sim}{v}\times B_1}{c}\right)\cdot\frac{\partial f_2^{*\pm}}{\partial \underset{\sim}{v}}+\left(E_2^*+\frac{\underset{\sim}{v}\times B_2^*}{c}\right)\cdot\frac{\partial f_1^{\pm}}{\partial \underset{\sim}{v}}\Big]$$

Since along the zeroeth-order trajectory

$$\left(\frac{d f_3^{\pm}}{dt}\right)_0=\frac{\partial f_3^{\pm}}{\partial t}+\underset{\sim}{v}\cdot\frac{\partial f_3^{\pm}}{\partial \underset{\sim}{r}}+\frac{ze\epsilon}{m_\pm}\left(\frac{\underset{\sim}{v}\times \underset{\sim}{B}_0}{c}\right)\cdot\frac{\partial f_3^{\pm}}{\partial \underset{\sim}{v}} \tag{2.13}$$

then

$$\left(\frac{d f_3^{\pm}}{dt}\right)_0=-\frac{ze\epsilon}{m_\pm}\left[\left(E_3+\frac{\underset{\sim}{v}\times \underset{\sim}{B}_3}{c}\right)\cdot\frac{\partial f_0^{\pm}}{\partial \underset{\sim}{v}}+\left(E_1+\frac{\underset{\sim}{v}\times \underset{\sim}{B}_1}{c}\right)\cdot\frac{\partial f_2^{*\pm}}{\partial \underset{\sim}{v}}+\left(E_2^*+\frac{\underset{\sim}{v}\times \underset{\sim}{B}_2^*}{c}\right)\cdot\frac{\partial f_1^{\pm}}{\partial \underset{\sim}{v}}\right] \tag{2.14}$$

By integrating equation 2.14 along the zeroeth-order trajectory

$$\begin{aligned} f_3^{\pm}=&-\frac{ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(E_3'+\frac{\underset{\sim}{v}'\times \underset{\sim}{B}_3'}{c}\right)\cdot\frac{\partial f_0^{\pm\prime}}{\partial \underset{\sim}{v}'}\,dt' \\ &-\frac{ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(E_1'+\frac{\underset{\sim}{v}'\times \underset{\sim}{B}_1'}{c}\right)\cdot\frac{\partial f_2^{*\pm\prime}}{\partial \underset{\sim}{v}'}\,dt' \\ &-\frac{ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(E_2^{*\prime}+\frac{\underset{\sim}{v}'\times \underset{\sim}{B}_2^{*\prime}}{c}\right)\cdot\frac{\partial f_1^{\pm\prime}}{\partial \underset{\sim}{v}'}\,dt' \end{aligned} \tag{2.15}$$

$f_3^{\pm}$ can then be written as

$$f_3^{\pm} = f_{33}^{\pm} + f_{32}^{\pm} + f_{31}^{\pm} \qquad (2.16)$$

where

$$f_{33}^{\pm} = -\frac{Ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(\underset{\sim}{E}_3' + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_3'}{c}\right)\cdot\frac{\partial f_0^{\pm'}}{\partial\underset{\sim}{v}'}\,dt' \qquad (2.17)$$

$$f_{32}^{\pm} = -\frac{Ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(\underset{\sim}{E}_1' + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_1'}{c}\right)\cdot\frac{\partial f_2^{\pm'*}}{\partial\underset{\sim}{v}'}\,dt' \qquad (2.18)$$

$$f_{31}^{\pm} = -\frac{Ze\epsilon}{m_\pm}\int_{-\infty}^{t}\left(E_2'^{*} + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_2'^{*}}{c}\right)\cdot\frac{\partial f_1^{\pm'}}{\partial\underset{\sim}{v}'}\,dt' \qquad (2.19)$$

Note: Specific reference to the species of the particle will be dropped at this point since necessary signs and factors are conserved by terms like $m, \epsilon, Z$.

## 2.3 Evaluation of $f_{31}$

To evaluate $f_{31}$ an expression for $f_1'$ is required. This is obtained by using the Fourier analysed Maxwell's equation

$$\underset{\sim}{K}_1\times\underset{\sim}{E}_1 = \frac{\omega_1}{c}\underset{\sim}{B}_1 \qquad (2.20)$$

to replace $\underset{\sim}{B}_1$ by $\underset{\sim}{E}_1$ in 2.10 to get

$$f_1' = -\frac{Ze\epsilon}{m_\pm}\int_{-\infty}^{t'}dt''\,\underset{\sim}{E}_1''\left(1 + \frac{\underset{\sim}{v}''\cdot\underset{\sim}{K}_1}{\omega_1} - \frac{\underset{\sim}{v}''\cdot\underset{\sim}{K}_1}{\omega_1}\right)\cdot\frac{\partial f_0''}{\partial\underset{\sim}{v}'} \qquad (2.21)$$

For Maxwellian, 2.21 reduces to

$$f_1' = -\frac{Ze\epsilon}{m}\int_{-\infty}^{t'}dt''\left(\underset{\sim}{E}_1''\cdot\underset{\sim}{v}''\right)\,f_0''(v) \qquad (2.22)$$

The integrand of 2.22 must be evaluated along the zeroeth-order trajectory defined by $\underset{\sim}{r} = \underset{\sim}{r}(t)$ . The solution to 2.6 which reaches $\underset{\sim}{V}'' = V'$ at $t'' = t'$ is

$$
\begin{aligned}
V_x'' &= V_x' \cos\Omega(t'-t'') - \epsilon V_y' \sin\Omega(t'-t'') \\
V_y'' &= \epsilon V_x' \sin\Omega(t'-t'') + V_y' \cos\Omega(t'-t'') \\
V_z'' &= V_z'
\end{aligned}
\tag{2.23}
$$

The integration of 2.23 yields for $\underset{\sim}{r}'' = \underset{\sim}{r}'$ at $t'' = t'$

$$
\begin{aligned}
x'' &= -\frac{V_x'}{\Omega} \sin\Omega(t'-t'') + \frac{\epsilon V_y'}{\Omega}\left[1 - \cos\Omega(t'-t'')\right] + x' \\
y'' &= -\frac{\epsilon V_x'}{\Omega}\left[1 - \cos\Omega(t'-t'')\right] - \frac{V_y'}{\Omega} \sin\Omega(t'-t'') + y' \\
z'' &= -V_z'(t'-t'') + z'
\end{aligned}
\tag{2.24}
$$

By using 2.23 and 2.24 to replace all doubled primed quantities by primed quantities

$$
f_1' = \frac{ze\epsilon}{m} e^{i(K_1 \cdot r' - \omega_1 t')} f_0(v') \int_0^\infty e^{i(a_1 V_x' + b_1 V_y' + c_1 V_z') + i\omega_1 \tau} \Big\{ (V_x' \cos\Omega\tau - \epsilon V_y' \sin\Omega\tau) E_{1x} + E_{1z} V_z' \Big\} d\tau
\tag{2.25}
$$

Where

$$
a_1 = -\frac{K_{x_1}}{\Omega} \sin\Omega\tau
\tag{2.26}
$$

$$
\tau = t' - t''
$$

$$b_1 = \frac{K_{x_1}}{\Omega}\left[1 - \cos\Omega\tau\right] \tag{2.27}$$

$$c_1 = -K_{z_1}\tau \tag{2.28}$$

By using equations 2.20 to express the magnetic field in terms of the electric field in equation 2.19

$$f_{31} = -\frac{zee}{m}\int_{-\infty}^{t} e^{i(K_2\cdot r' - \omega_2 t')}\left\{E_{2x}^{*}\left(1 - \frac{K_{2z}V_z'}{\omega_2}\right)\frac{\partial f_1'}{\partial V_x'} + \frac{E_{2x}^{*}V_x'K_{2z}}{\omega_2}\frac{\partial f_1'}{\partial V_z'}\right\}dt' \tag{2.29}$$

and using 2.25 for $f_1'$

$$f_{31} = -\frac{z^2e^2E_{1x}E_{2x}^{*}}{mKT}\int_{-\infty}^{t}dt'\int_{0}^{\infty}d\tau\, e^{i(K\cdot r' - \omega t') + i(a_1V_x' + b_1V_y' + c_1V_z' + \omega_1\tau)} f_0(v')\left\{\left(V_x'\cos\Omega\tau - \epsilon V_y'\sin\Omega\tau\right.\right.$$

$$\left.+ \frac{E_{1z}V_z'}{E_{1x}}\right)\left(\left(1 - \frac{K_{2z}V_z'}{\omega_2}\right)\left(ia_1 - \frac{mV_x'}{KT}\right) + \frac{V_x'K_{2z}}{\omega_2}\left(ic_1 - \frac{mV_z'}{KT}\right)\right) \tag{2.30}$$

$$\left. + \left(1 - \frac{K_{2z}V_z'}{\omega_2}\right)\cos\Omega\tau + \frac{E_{1z}V_x'K_{2z}}{E_{1x}\,\omega_2}\right\}$$

The integration with respect to $t'$ must be calculated along the zeroeth-order trajectory defined by $\underset{\sim}{r} = \underset{\sim}{r}(t)$. The solution which reaches $\underset{\sim}{V}'' = \underset{\sim}{V}'$ & $\underset{\sim}{r}'' = \underset{\sim}{r}'$ at $t'' = t'$ is obtained from 2.23 and 2.24 by replacing $t'$ by $t''$ and $t$ by $t'$. Employing these new equations and substituting $\tau'$ for $t' - t''$ yields after considerable algebra:

$$f_{31} = \frac{z^2 e^2 E_{1x} E_{2x}^*}{mKT} \int_0^{\infty}\int_0^{\infty} f_0(v)\, e^{i(aV_x + bV_y + (\omega_1 - K_{1z}V_z)\tau + (\omega - K_zV_z)\tau')} \Big\{ \beta_{11} + \beta_{12}V_x + \beta_{13}V_y \tag{2.31}$$

$$+ \beta_{14}V_x^2 + \beta_{15}V_y^2 + \beta_{16}V_xV_y \Big\} d\tau\, d\tau'$$

The $\beta's$ can be expressed in the more useful form (using exponential notation)

$$\beta_{11} = \eta_{11} e^{i\Omega\tau'' - i\Omega\tau'} + \eta_{12} e^{-i\Omega\tau'' + i\Omega\tau'}$$

$$\beta_{12} = e^{i\Omega\tau'}\left(\eta_{13}\left(1 + e^{-i\Omega 2\tau''}\right) - \eta_{14}\right) - e^{-i\Omega\tau'}\left(\eta_{13}\left(1 + e^{2i\Omega\tau''}\right) + \eta_{14}\right)$$

$$\beta_{13} = e^{-i\epsilon\Omega\tau'}\left(i\eta_{13} e^{i\epsilon\Omega\tau''}\left(e^{-i\Omega\tau''} - e^{i\Omega\tau''}\right) + i\eta_{14}\right)$$

$$+ e^{i\epsilon\Omega\tau'}\left(i\eta_{13} e^{-i\epsilon\Omega\tau''}\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right) - i\eta_{14}\right)$$

$$\beta_{14} = \eta_{15}\left( e^{i\Omega\tau'}\left(e^{i\Omega\tau''} + e^{-i\Omega\tau''}\right) + e^{-i\Omega\tau'}\left(e^{i\Omega\tau''} + e^{-i\Omega\tau''}\right)\right) \tag{2.31A}$$

$$\beta_{15} = \eta_{15}\left( e^{-i\Omega\tau'}\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right) - e^{i\Omega\tau'}\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right)\right)$$

$$\beta_{16} = 2i\eta_{15}\left( e^{i\epsilon\Omega\tau'' + i\epsilon\Omega\tau'} - e^{-i\epsilon\Omega\tau'' - i\epsilon\Omega\tau'}\right)$$

where

$$\eta_{11} = \frac{1}{2}\left(1 - \frac{V_z K_z}{\omega_2}\right)\left(\frac{E_{1z} V_z K_{x_1}}{E_{1x}\Omega} - 1\right)$$

$$\eta_{12} = -\frac{1}{2}\left(1 - \frac{V_z K_z}{\omega_2}\right)\left(\frac{E_{1z} V_z K_{x_1}}{E_{1x}\Omega} + 1\right)$$

$$\eta_{13} = -\left(1 - \frac{V_z K_z}{\omega_2}\right)\left(\frac{K_{x_1}}{4\Omega}\right)$$

$$\eta_{14} = \frac{E_{1z}}{2E_{1x}}\left(\frac{K_z}{\omega_2} - V_z\left(i\frac{K_z K_{z_1}}{\omega_2}\tau + \frac{m}{KT}\right)\right)$$

$$\eta_{15} = \frac{1}{4}\left(\frac{m}{KT} + i\frac{K_z K_{z_1}}{\omega_2}\tau\right)$$

$$a = -\frac{K_x}{\Omega}\operatorname{Sin}\Omega(\tau+\tau')$$

$$b = \frac{\epsilon K_x}{\Omega}\left[1 - \operatorname{Cos}\Omega(\tau+\tau')\right] \tag{2.31B}$$

## 2.4 Velocity Moments of $f_{31}$

In the solution to this problem, the quantities of interest are the zeroeth and first moments of our distribution over velocity space. These moments are of the form:

$$\langle g(v) f_{31} \rangle_{\perp} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(v)\, f_{31}(v)\, dV_x\, dV_y \tag{2.32}$$

The calculation of these moments involves integrals of the form

$$G(p,a) = \left(\frac{m}{2\pi KT}\right)^{1/2} \int_{-\infty}^{\infty} v_r^p \, e^{\left(iav - \frac{mv^2}{2KT}\right)} dv \tag{2.33}$$

which can be evaluated by completing the square to yield

$$G(0,a) = e^{-\frac{a^2 KT}{2m}}$$

$$G(1,a) = \frac{iaKT}{m} e^{-\frac{a^2 KT}{2m}}$$

$$G(2,a) = \left(\frac{KT}{m} - \frac{a^2 K^2 T^2}{m^2}\right) e^{-\frac{a^2 KT}{2m}} \tag{2.34}$$

$$G(3,a) = i\frac{aKT}{m}\left(\frac{3KT}{m} - \frac{a^2 K^2 T^2}{m^2}\right) e^{-\frac{a^2 K^2 T^2}{2m}}$$

In carrying out the integration over $v_x$ and $v_y$ one obtains terms in the exponent of the form

$$-\frac{(a^2+b^2)}{2}\frac{KT}{m} \tag{2.35}$$

which can be evaluated to yield

$$-\frac{(a^2+b^2)}{2}\frac{KT}{m} = -\lambda\left(1 - \cos\Omega(\tau+\tau')\right) \tag{2.36}$$

where

$$\lambda = \frac{K_x^2 KT}{\Omega^2 m}$$

The quantity $\lambda$ which is the ratio of the square of the Larmor radius to the perpendicular wavelength is an important expansion parameter. In this thesis, consideration will be given mainly to the case where $\lambda$ is large. The regime of large $\lambda$ corresponds to a plasma where the perpendicular wavelength of the difference frequency is much smaller than the electron Larmor radius. With the help of 2.34, one can evaluate the zeroeth and first moments of $f_{31}$.

$$\langle f_{31} \rangle_{\perp} = \frac{z^2 e^2 E_{1x} E_{2x}^*}{mkT} e^{i(\underline{k}\cdot\underline{r}-\omega t)} \iint_0^\infty e^{i\psi-\lambda(1-\cos\Omega(\tau+\tau'))} f_0(v_z) \Big\{ B_{11} \tag{2.37}$$

$$+ B_{12}\, i\frac{akT}{m} + B_{13}\, i\frac{bkT}{m} + B_{14}\left(\frac{kT}{m} - \frac{a^2k^2T^2}{m^2}\right) + B_{15}\left(\frac{kT}{m} - \frac{b^2k^2T^2}{m^2}\right)$$

$$- B_{16}\, ab\frac{k^2T^2}{m^2}\Big\}\, d\tau\, d\tau'$$

$$\langle f_{31} v_y \rangle_{\perp} = \frac{z^2 e^2 E_{1x} E_{2x}^*}{mkT} e^{i(\underline{k}\cdot\underline{r}-\omega t)} \iint_0^\infty f_0(v_z) \tag{2.38}$$

$$e^{i\psi-\lambda(1-\cos\Omega(\tau+\tau'))} \Big\{ B_{11}\, i\frac{bkT}{m} - B_{12}\, ab\frac{k^2T^2}{m^2} + B_{13}\left(\frac{kT}{m} - \frac{b^2k^2T^2}{m^2}\right) + B_{14}\, i\frac{bkT}{m}$$

$$\left(\frac{kT}{m} - \frac{a^2k^2T^2}{m^2}\right) + B_{15}\, i\frac{bkT}{m}\left(\frac{3kT}{m} - \frac{b^2k^2T^2}{m^2}\right) + B_{16}\, i\frac{akT}{m}\left(\frac{kT}{m} - \frac{b^2k^2T^2}{m^2}\right)\Big\}\, d\tau\, d\tau'$$

$$\langle f_{31} v_x \rangle_\perp = \frac{z^2 e^2 E_{1x} E_{2x}^{*} e^{i(\underline{k}\cdot\underline{r}-\omega t)}}{m K T} \int_0^\infty\!\!\int_0^\infty e^{i\psi - \lambda(1-\cos\Omega(\tau+\tau'))} f_0(v_z) \Big\{ \beta_{11} \frac{i a K T}{m}$$

$$+ \beta_{12}\left(\frac{KT}{m} - \frac{a^2K^2T^2}{m^2}\right) - \beta_{13}\frac{abK^2T^2}{m^2} + \beta_{14}\frac{iaKT}{m}\left(\frac{3KT}{m} - \frac{a^2K^2T^2}{m^2}\right)$$

(2.39)

$$+ \beta_{15}\frac{iaKT}{m}\left(\frac{KT}{m} - \frac{b^2K^2T^2}{m^2}\right) + \beta_{16}\frac{ibKT}{m}\left(\frac{KT}{m} - \frac{a^2K^2T^2}{m^2}\right)\Big\} d\tau\, d\tau'$$

$$\psi = (\omega - K_z v_z)\tau' + (\omega_1 - K_{1z} v_z)\tau$$

the term $e^{\lambda\cos\Omega(\tau+\tau')}$ can be expanded by using the Bessel function equality

$$e^{\lambda\cos\Omega(\tau+\tau')} = \sum_{n=-\infty}^{\infty} I_n(\lambda)\, e^{in\Omega(\tau+\tau')} \qquad (2.40)$$

To perform the integration of $\tau, \tau'$ and $V_z$ , one must collect common exponential terms which means that terms like $a, b, \ldots\ldots$ must be expanded.

$$\frac{a\,kT}{m} = i\alpha\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right)$$

$$a^2\left(\frac{kT}{m}\right)^2 = -\alpha^2\left(e^{i2\Omega\tau''} - 2 + e^{-i2\Omega\tau''}\right)$$

$$a^3\left(\frac{kT}{m}\right)^3 = -i\alpha^3\left(e^{i3\Omega\tau''} - 3e^{i\Omega\tau''} + 3e^{-i\Omega\tau''} - e^{-i3\Omega\tau''}\right)$$

$$b\left(\frac{kT}{m}\right) = \epsilon\alpha\left(2 - e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right)$$

$$b^2\left(\frac{kT}{m}\right)^2 = \alpha^2\left(6 - 4e^{i\Omega\tau''} - 4e^{-i\Omega\tau''} + e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right)$$

$$b^3\left(\frac{kT}{m}\right)^2 = \epsilon\alpha^3\Big(20 - 15\left(e^{i\Omega\tau''} + e^{-i\Omega\tau''}\right) + 6\left(e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right)$$

$$-\left(e^{i3\Omega\tau''} + e^{-i3\Omega\tau''}\right)\Big) \tag{2.41}$$

$$ab\left(\frac{kT}{m}\right)^2 = i\epsilon\alpha^2\left(2\left(e^{i\Omega\tau''} + e^{-i\Omega\tau''}\right) - e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right)$$

$$a^2b\left(\frac{kT}{m}\right)^3 = -\epsilon\alpha^3\Big(-4 + e^{i\Omega\tau''} + e^{-i\Omega\tau''} + 2\left(e^{i2\Omega\tau''} + e^{-i2\Omega\tau''}\right)$$

$$- e^{i3\Omega\tau''} - e^{-i3\Omega\tau''}\Big)$$

$$ab^2\left(\frac{kT}{m}\right)^3 = i\alpha^3\Big(5\left(e^{i\Omega\tau''} - e^{-i\Omega\tau''}\right) - 4\left(e^{i2\Omega\tau''} - e^{-i\Omega\tau''}\right)$$

$$+ e^{i3\Omega\tau''} - e^{-i3\Omega\tau''}\Big)$$

where $$\alpha = \frac{K_x^2 KT}{2\Omega m}$$

$$\tau'' = \tau + \tau'$$

Using the set of equations 2.41 for the $a$'s, $b$'s etc., 2.40 for the expansion of $e^{\lambda \cos\Omega(\tau+\tau')}$, equations 2.37-2.39 can be written as

$$\langle f_{31}\rangle_1 = \frac{Z^2 e^2 E_{1x} E_{2x}^* e^{i(\underline{k}\cdot\underline{r}-\omega t)}}{mKT} \sum_{n=-\infty}^{\infty} \iint_0^\infty e^{i\psi + i\Omega n\tau''} \Big\{ \eta_{12} I_{n+1}(\lambda) e^{i\Omega\tau'}$$

$$+ \eta_{12} I_{n-1} e^{-i\Omega\tau'} + \alpha\eta_{13}\Big( (I_{n+3} - I_{n-1}) e^{i\Omega\tau'} + (I_{n-3} - I_{n+1}) e^{-i\Omega\tau'}$$

$$- \epsilon\Big(\Big( 2(I_{n-1+\epsilon} - I_{n+1+\epsilon}) - I_{n-2+\epsilon} + I_{n+2+\epsilon}\Big) e^{i\epsilon\Omega\tau'}$$

$$+ \Big( 2( I_{n+1-\epsilon} - I_{n-1-\epsilon}) + I_{n-2-\epsilon} - I_{n+2-\epsilon}\Big) e^{-i\epsilon\Omega\tau'}\Big)\Big) + \alpha\eta_{14}\Big($$

$$(I_{n-1} - I_{n+1}) e^{i\Omega\tau'} + (I_{n-1} - I_{n+1}) e^{-i\Omega\tau'} + \epsilon\Big(\Big( 2 I_n - I_{n+1} - I_{n-1}\Big) e^{i\epsilon\Omega\tau'}$$

$$- \Big( 2 I_n - I_{n+1} - I_{n-1}\Big) e^{-i\epsilon\Omega\tau'} - \eta_{15}\alpha^2 \Big(\Big(\Big( \frac{2KT}{m\alpha^2} - 6\Big) I_{n+1}$$

$$+ 4( I_{n+2} - I_{n-2} + I_{n-1}) + 2 I_{n-3}\Big) e^{i\Omega\tau'} + \Big(\Big( \frac{2KT}{m\alpha^2} - 6\Big) I_{n-1}$$

$$+ 4( I_{n-2} - I_{n+2} + I_{n+1}) + 2 I_{n+3}\Big) e^{-i\Omega\tau'} + 2\epsilon\Big(\Big( 2( I_{n-1-\epsilon} - I_{n+1-\epsilon})$$

$$- I_{n-2-\epsilon} + I_{n+2-\epsilon}\Big) e^{i\epsilon\Omega\tau'} - \Big( 2( I_{n-1+\epsilon} - I_{n+1-\epsilon}) - I_{n-2+\epsilon} + I_{n+2+\epsilon}\Big)$$

$$e^{-i\epsilon\Omega\tau'}\Big)\Big)\Big\} d\tau\, d\tau'$$

$$\langle f_{31} v_x \rangle_1 = \frac{z^2 e^2 E_{1x} E_{2x}^* e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)-\lambda}}{mkT} \sum_{n=-\infty}^{\infty} \iint_0^{\infty} e^{i\psi + in\Omega\underset{\sim}{\tau}''} d\tau\, d\tau'\, f_0(v_z)$$

$$\Big\{ -\eta_{11}\alpha(I_{n-2}-I_n)e^{-i\Omega\tau'} - \eta_{12}\alpha(I_n - I_{n+2})e^{i\Omega\tau'} + \eta_{13}\alpha^2\Big(\Big(\frac{kT}{m\alpha^2}$$

$$(I_{n+2}+I_n) + I_{n-2} - I_{n+2} + I_{n+4} - I_n\Big)e^{i\Omega\tau'} - \Big(\frac{kT}{m\alpha^2}(I_{n-2}+I_n)$$

$$+ I_{n+2} - I_{n-2} + I_{n-4} - I_n\Big)e^{-i\Omega\tau'} + \epsilon\Big(\Big(2(I_{n-2+\epsilon} + I_{n+2+\epsilon}) - 4I_{n+\epsilon}$$

$$- I_{n-3+\epsilon} - I_{n+3+\epsilon} + I_{n-1+\epsilon} + I_{n+1+\epsilon}\Big)e^{i\epsilon\Omega\tau'} + \Big(4I_{n-\epsilon} - 2(I_{n-2-\epsilon}$$

$$+ I_{n+2-\epsilon}) + I_{n-3-\epsilon} - I_{n+3-\epsilon} - I_{n-1-\epsilon} - I_{n+1-\epsilon}\Big)e^{-i\epsilon\Omega\tau'}\Big)\Big)$$

$$+ \eta_{14}\alpha^2\Big(\Big(\frac{kT}{m\alpha^2} I_n - 2I_n + I_{n+2} + I_{n-2}\Big)e^{i\Omega\tau'} - \Big(\frac{kT}{m\alpha^2} I_n - 2I_n$$

$$+ I_{n-2} + I_{n+2}\Big)e^{-i\Omega\tau'} - \epsilon\Big(\Big(2(I_{n-1} - I_{n+1}) - I_{n-2} + I_{n+2}\Big)e^{i\epsilon\Omega\tau'}$$

$$-\Big(2(I_{n-1} - I_{n+1}) - I_{n-2} + I_{n+2}\Big)e^{-i\epsilon\Omega\tau'}\Big)\Big) + \eta_{15}\alpha^2\Big(\Big(\frac{2kT}{m\alpha^2}(I_n$$

$$+ I_{n-2} - 2I_{n+2}) + 2I_{n-2} - 10I_n + 6I_{n+2} + 2I_{n-4} + 4(I_{n-1} + I_{n+1} - I_{n-3}$$

$$- I_{n+3})\Big)e^{i\Omega\tau'} + \Big(\frac{2kT}{m\alpha^2}(2I_{n-2} - I_n - I_{n+2}) + 10I_n - 6I_{n-2}$$

$$+ 4(I_{n+3} + I_{n-3}) - 4(I_{n+1} + I_{n-1}) - 2(I_{n+2} + I_{n+4})\Big)e^{-i\Omega\tau'}$$

$$+\epsilon\Big(\Big(\frac{2kT}{m\alpha^2}\big(2I_{n-\epsilon}-I_{n-1-\epsilon}-I_{n+1-\epsilon}\big)-4I_{n-\epsilon}+I_{n-1-\epsilon}+I_{n+1-\epsilon}$$

$$+2I_{n-2-\epsilon}+2I_{n+2-\epsilon}-I_{n-3-\epsilon}-I_{n+3-\epsilon}\Big)e^{i\epsilon\Omega\tau'}+e^{-i\epsilon\Omega\tau'}\Big($$

$$-\frac{2kT}{m\alpha^2}\big(2I_{n+\epsilon}-I_{n-1+\epsilon}-I_{n+1+\epsilon}\big)-4I_{n+\epsilon}+I_{n-1+\epsilon}+I_{n+1+\epsilon}$$

$$+2\big(I_{n-2+\epsilon}+I_{n+2+\epsilon}\big)-I_{n+3+\epsilon}-I_{n-3+\epsilon}\Big)\Big)\Big\}\,.$$

$$\langle f_{31}v_y\rangle=\frac{z^2e^2E_{1x}E_{2x}^{*}e^{i(\underline{k}\cdot\underline{r}-\omega t)-\lambda}}{mkT}\sum_{n=-\infty}^{\infty}\iint_0^{\infty}e^{i\psi+in\Omega\tau''}\Big\{i\epsilon\alpha\eta_{11}\big(2I_{n-1}$$

$$-I_{n-2}-I_n\big)e^{-i\Omega\tau'}+i\epsilon\alpha\eta_{12}e^{i\Omega\tau'}\big(2I_{n+1}-I_n-I_{n+2}\big)+i\epsilon\alpha^2\eta_{13}$$

$$\Big(\Big(2(I_{n+3}-I_{n-1})+I_{n-2}-I_{n+2}+I_n-I_{n+4}\Big)e^{i\Omega\tau'}+\Big(I_n-2I_{n+1}$$

$$-I_{n-2}+I_{n+2}+2I_{n-3}-I_{n-4}\Big)e^{-i\Omega\tau'}+\Big(\frac{kT}{m\alpha^2}\big(I_{n-1+\epsilon}-I_{n+1+\epsilon}\big)$$

$$-5\big(I_{n-1+\epsilon}-I_{n+1+\epsilon}\big)+4\big(I_{n-2+\epsilon}-I_{n+2+\epsilon}\big)-I_{n-3+\epsilon}+I_{n+3+\epsilon}\Big)e^{i\epsilon\Omega\tau'}$$

$$+\Big(\frac{kT}{m\alpha^2}\big(I_{n+1-\epsilon}-I_{n-1-\epsilon}\big)-5\big(I_{n+1-\epsilon}-I_{n-1-\epsilon}\big)-4\big(I_{n-2-\epsilon}-I_{n+1-\epsilon}\big)$$

$$+I_{n-3-\epsilon}-I_{n+3-\epsilon}\Big)e^{-i\epsilon\Omega\tau'}\Big)+i\epsilon\alpha^2\eta_{14}\Big(\Big(2\big(I_{n-1}-I_{n+1}\big)$$

$$-I_{n-2}+I_{n+2}\Big)e^{i\Omega\tau'}+\Big(2\big(I_{n-1}-I_{n+1}\big)-I_{n+2}+I_{n-2}\Big)e^{-i\Omega\tau'}$$

$$-\epsilon\Big(\Big(\frac{kT}{m\alpha^2}-6\Big)I_n+4\big(I_{n-1}+I_{n+1}\big)-I_{n+2}-I_{n-2}\Big)e^{i\epsilon\Omega\tau'}$$

$$+\epsilon\Big(\big(\frac{kT}{m\alpha^2}-6\big)I_n+4(I_{n-1}+I_{n+1})-I_{n+2}-I_{n-2}\Big)e^{-i\epsilon\Omega\tau'}$$

$$+i\epsilon\alpha^3\eta_{15}\Big(\Big(-\frac{2kT}{m\alpha^2}(4I_{n+1}-2I_{n-1}+I_{n-2}-2I_{n+2}-I_n)-12I_{n-1}$$

$$+14(I_{n-2}-I_{n+2})-2I_n-8I_{n-3}+4I_{n+3}+16I_{n+1}+2I_{n-4}\Big)e^{i\Omega\tau'}$$

$$+\Big(-\frac{2kT}{m\alpha^2}(4I_{n-1}-2I_{n-2}-I_n-2I_{n+1}+I_{n+2})+16I_{n-1}-14I_{n-2}$$

$$-2I_n+4I_{n-3}-12I_{n+1}+14I_{n+2}-8I_{n+3}+2I_{n+4}\Big)e^{-i\Omega\tau'}$$

$$+\epsilon\Big(\frac{kT}{m\alpha^2}(I_{n-1-\epsilon}-I_{n+1-\epsilon})-5(I_{n-1-\epsilon}-I_{n+1-\epsilon})+4(I_{n-2-\epsilon}-I_{n+2-\epsilon})$$

$$-I_{n-3-\epsilon}+I_{n+3-\epsilon}\Big)e^{i\epsilon\Omega\tau'}+\epsilon\Big(-\frac{kT}{m\alpha^2}(I_{n-1+\epsilon}-I_{n+1+\epsilon})+5(I_{n-1+\epsilon}$$

$$-I_{n+1+\epsilon})-4(I_{n-2+\epsilon}-I_{n+2+\epsilon})+I_{n-3+\epsilon}-I_{n+3+\epsilon}\Big)e^{-i\epsilon\Omega\tau'}\Big)\Big\}f_0(v_z)\,d\tau\,d\tau'$$

The integration with respect to $\tau$ can now be performed in equations 2.42–2.44. Two types of integrals involving $\tau$ are encountered in equations 2.42–2.44.

1. Integrals of the Form

$$\int_0^\infty d\tau\,(\quad)\,e^{i(n\Omega+\omega_1-K_{1z}V_z)\tau}$$

where $(\quad)$ is independent of $\tau$. If $\omega_1$ is assumed to have a small positive imaginary component then this integral can be evaluated to give

$$\frac{i(\quad)}{n\Omega+\omega_1-K_{z1}V_z}$$

2. Integrals of the Form

$$\int_0^\infty d\tau\,(\quad)\,\tau\, e^{i(n\Omega+\omega_1-K_{1z}V_z)\tau}$$

Integration by parts give this integral the value

$$\frac{-1}{(n\Omega+\omega_1+K_{1z}V_z)^2}$$

The results in 1 and 2 can be expanded to give

$$\frac{i(\quad)}{n\Omega+\omega_1+K_{1z}V_z} = \frac{i(\quad)}{n\Omega+\omega_1}\left\{1+\frac{K_{1z}V_z}{n\Omega+\omega_1}+\cdots\cdots\right\} \tag{2.45}$$

and

$$\frac{-1}{(n\Omega+\omega_1+K_{1z}V_z)^2} = \frac{-1}{(n\Omega+\omega_1)^2}\left\{1+\frac{2K_{1z}V_z}{n\Omega+\omega_1}+\cdots\cdots\right\} \tag{2.46}$$

This expansion is valid providing $\omega_1$ is always sufficiently removed from the cyclotron frequencies of the ions of electrons. The approximation $|K_{1z}V_z| << |n\Omega+\omega_1|$ is equivalent to the assumption that the thermal velocity over the doppler-shifted phase velocity is much less than 1. By integrating 2.42-2.44, rearranging the terms in summations and finally collecting terms of common power in $V_z$.

$$\langle f_{31}\rangle_\perp = \frac{Z^2e^2E_{1x}E_{2x}^*}{mKT}\, e^{i(\underline{K}\cdot\underline{r}-\omega t)-\lambda}\sum_{n=-\infty}^{\infty}\int_0^\infty f_0(V_z)\,d\tau' \tag{2.47}$$

$$\left\{e^{i(\omega+n\Omega+K_zV_z)\tau'}\left(\phi_{11}+\phi_{12}V_z+\phi_{13}V_z^2\right)\right\}$$

$$\langle f_{31}V_x\rangle_\perp = \frac{Z^2e^2E_{1x}E_{2x}^*}{mKT}\, e^{i(\underline{K}\cdot\underline{r}-\omega t)}\sum_{n=-\infty}^{\infty}\iint_0^\infty f_0(V_z)\,d\tau' \tag{2.48}$$

$$\left\{e^{i(\omega+n\Omega-K_{1z}V_z)\tau'}\left(\phi_{21}+\phi_{22}V_z+\phi_{23}V_z^2\right)\right\}$$

$$\langle f_{31} v_y \rangle_\perp = \frac{z^2 e^2 E_{1x} E_{2x}^*}{mKT} e^{i(\underline{k}\cdot\underline{r}-\omega t)-\lambda} \sum_{n=-\infty}^{\infty} \int_0^\infty f(v_z) e^{i(n\Omega+\omega-k_z v_z)\tau'} \left\{ \phi_{31} + \phi_{32} v_z + \phi_{33} v_z^2 \right\} d\tau' \tag{2.49}$$

where

$$\phi_{11} = 2\xi_{11} I_n + \xi_{51}\gamma_{11} + 2\epsilon\xi_{31}(I_{n-2} - I_{n+2}) - \xi_{41}\gamma_{12}$$

$$\phi_{12} = \xi_{21} I_n + \xi_{52}\gamma_{11} + 2\epsilon\xi_{32}(I_{n-2} - I_{n+2}) - \xi_{42}\gamma_{12}$$

$$\phi_{13} = \xi_{22} I_n + \xi_{53}\gamma_{11} + 2\epsilon\xi_{33}(I_{n-2} - I_{n+2})$$

$$\phi_{21} = -2\xi_{11}\alpha(I_{n-1} - I_{n+1}) + \xi_{51}\gamma_{21}\alpha + \xi_{41}\gamma_{22}\alpha$$

$$\phi_{22} = -\alpha\xi_{21}(I_{n-1} - I_{n+1}) + \alpha\xi_{52}\gamma_{21} + \xi_{41}\alpha\gamma_{22}$$

$$\phi_{23} = -\alpha\xi_{22}(I_{n-1} - I_{n+1}) + \xi_{53}\gamma_{21}\alpha + \xi_{42}\gamma_{22}\alpha$$

$$\phi_{31} = 2i\epsilon\alpha\xi_{11}(2I_n - I_{n-1} - I_{n+1}) + (\xi_{51}\gamma_{31} - \xi_{41}\gamma_{32})i\epsilon\alpha$$

$$\phi_{32} = i\epsilon\alpha\left(\xi_{21}(2I_n - I_{n-1} - I_{n+1}) + \xi_{52}\gamma_{31} - \xi_{42}\gamma_{32}\right.$$

$$\phi_{33} = i\epsilon\alpha\xi_{22}(2I_n - I_{n-1} - I_{n+1}) + \xi_{53}\gamma_{31}$$

$$\xi_{11} = \frac{-i}{n\Omega + \omega_1}$$

$$\xi_{21} = \frac{-2i}{n\Omega+\omega_1}\left\{\frac{K_{z1}}{n\Omega+\omega_1} - \frac{K_z}{\omega_2}\right\}$$

$$\xi_{22} = \frac{2iK_zK_{z_1}}{(n\Omega+\omega_1)^2\,\omega_2}$$

$$\xi_{31} = -\frac{i\lambda}{8(n\Omega+\omega_1)} \qquad \xi_{32} = -\frac{i\lambda}{8(n\Omega+\omega_1)}\left\{\frac{K_{z_1}}{n\Omega+\omega_1} - \frac{K_z}{\omega_2}\right\}$$

$$\xi_{33} = \frac{i\lambda}{8(n\Omega+\omega_1)^2}\frac{K_{z_1}K_z}{\omega_2}$$

$$\xi_{41} = \frac{-i\alpha^2}{4(n\Omega+\omega_1)}\left\{\frac{m}{KT} - \frac{K_zK_{z1}}{\omega_2(n\Omega+\omega_1)}\right\}$$

$$\xi_{42} = \frac{-i\alpha^2K_{z1}}{4(n\Omega+\omega_1)^2}\left\{\frac{m}{KT} - \frac{K_zK_{z_1}}{\omega_2(n\Omega+\omega_1)}\right\}$$

$$\xi_{51} = \frac{i\alpha E_{1z}K_z}{2E_{1x}\omega_2(n\Omega+\omega_1)}$$

$$\xi_{52} = -\frac{i\alpha E_{1z}}{2E_{1x}(n\Omega+\omega_1)}\left(\frac{m}{KT} - \frac{2K_{z_1}K_z}{\omega_2(n\Omega+\omega_1)}\right)$$

$$\xi_{53} = -\xi_{52}\frac{K_{z_1}}{(n\Omega+\omega_1)}$$

$$\gamma_{11} = I_{n-2} - I_{n+2} + \epsilon\Big((I_{n-\epsilon} - I_{n+\epsilon})\times 2 - I_{n+1-\epsilon}$$

$$- I_{n-1-\epsilon} + I_{n+1+\epsilon} + I_{n-1+\epsilon}\Big)$$

$$\gamma_{12} = 2\left(\frac{2kT}{m\alpha^2} - 6\right)I_n + 4\left(I_{n+1} + I_{n-1} + I_{n-2} + I_{n+2}\right.$$

$$- I_{n+3} - I_{n-3} + 2(I_{n-4} + I_{n+4}) + 2\epsilon\left( 2(I_{n-1-2\epsilon}\right.$$

$$- I_{n+1+2\epsilon}) - I_{n-2-2\epsilon} + I_{n+2-2\epsilon} - 2(I_{n-1+2\epsilon} - I_{n+1+2\epsilon})$$

$$\left. + I_{n-2+2\epsilon} - I_{n+2+2\epsilon}\right)$$

$$\gamma_{21} = -\left( \frac{kT}{m\alpha^2}( I_{n-1} + I_{n+1}) + I_{n-3} + I_{n+3} - I_{n-1} - I_{n+1} + \epsilon(2(I_{n-1-\epsilon}\right.$$

$$- I_{n+1-\epsilon}) - I_{n-2-\epsilon} + I_{n+2-\epsilon} - 2(I_{n-1+\epsilon} - I_{n+1+\epsilon})$$

$$\left. + I_{n-2+\epsilon} - I_{n+2+\epsilon}\right)$$

$$\gamma_{22} = \frac{2kT}{m\alpha^2}(I_{n-3} - I_{n-1} - 2I_{n+1}) + 2I_{n-3} - 10I_{n-1} + 6I_{n+1}$$

$$+ 2I_{n-5} + 4(I_{n-2} + I_n - I_{n+2} - I_{n-4}) + \frac{2kT}{m\alpha^2}(2I_{n-1} - I_{n+1} - I_{n+3})$$

$$+ 10I_{n+1} - 6I_{n-1} + 4(I_{n+4} + I_{n-2} - I_{n+2} - I_n) - 2(I_{n+3} + I_{n+5})$$

$$+ 2\epsilon\left( \frac{kT}{m\alpha^2}(2I_{n-2\epsilon} - I_{n-1-2\epsilon} - I_{n+1-2\epsilon}) - 4I_{n-2\epsilon} + I_{n-1-2\epsilon}\right.$$

$$+ I_{n+1-2\epsilon} + 2(I_{n-2(1+\epsilon)} + I_{n+2(1-\epsilon)}) - I_{n-3-2\epsilon} - I_{n+3-2\epsilon}$$

$$- \frac{kT}{m\alpha^2}(2I_{n+2\epsilon} - I_{n-1+2\epsilon} - I_{n+1+2\epsilon}) + 4I_{n+2\epsilon} - I_{n+1+2\epsilon} - I_{n-1+2\epsilon}$$

$$- 2\left( I_{n-2(1-\epsilon)} + I_{n+2(1+\epsilon)}\right) + I_{n-3+2\epsilon} + I_{n+3+2\epsilon} \Big)$$

$$\gamma_{31} = 2(I_{n-2} - I_{n+2}) + I_{n+1} + I_{n-1} - I_{n+3} - I_{n-3} + \epsilon\Big( \frac{kT}{m\alpha^2}\big(I_{n+\epsilon}$$

$$- I_{n-\epsilon}\big) + 6(I_{n-\epsilon} - I_{n+\epsilon}) - 4(I_{n+1-\epsilon} + I_{n-1-\epsilon}) + I_{n-2-\epsilon}$$

$$+ I_{n+2-\epsilon} + 4(I_{n+1+\epsilon} + I_{n-1+\epsilon}) - I_{n+2+\epsilon} - I_{n-2+\epsilon}$$

$$\gamma_{32} = \frac{2kT}{m\alpha^2}\big( 2I_n - 2I_{n-2} + I_{n-3} - 2I_{n+1} - I_{n-1} + 4I_n - 2I_{n-1}$$

$$- I_{n+1} - 2I_{n+2} + I_{n+3}\big) + 12I_{n-2} - 14(I_{n-3} - I_{n+1}) + 2I_{n-1}$$

$$+ 8I_{n-4} - 4I_{n+2} - 16I_n - 2I_{n-5} + 14I_{n-1} - 16I_n + 2I_{n+1}$$

$$- 4I_{n-2} + 12I_{n+2} - 14I_{n+3} + 8I_{n+4} - 2I_{n+5} - 2\epsilon\Big(\frac{kT}{m\alpha^2}$$

$$\big(I_{n-1-2\epsilon} - I_{n+1+2\epsilon} - I_{n-1+2\epsilon} + I_{n+1-2\epsilon}\big) - 5\big(I_{n-1-2\epsilon} - I_{n+1-2\epsilon}$$

$$+ I_{n+1+2\epsilon} - I_{n-1+2\epsilon}\big) + 4\big(I_{n-2(1+\epsilon)} - I_{n+2(1-\epsilon)} - I_{n-2(1-\epsilon)}$$

$$+ I_{n+2(1+\epsilon)} + I_{n+3-2\epsilon} - I_{n-3-2\epsilon} + I_{n-3+2\epsilon} - I_{n+3+2\epsilon}$$

To complete the calculations of the second-order currents induces in the plasma by high frequency incident fields, the integration over $\tau'$ and $V_z$ must be performed. All of the integrals remaining to be evaluated are of the form

$$F_p = \frac{K_z}{\sqrt{\pi}} \int_{-\infty}^{\infty}\int_{0}^{\infty} V_z^{\,p}\, e^{i(\omega+n\Omega-K_zV_z)\tau' - \frac{mV_z^2}{2KT}}\, d\tau'\, dV_z$$

where $p = 0,1,2,3$

In order to demonstrate the convergence of $F_p$ the square in the exponent may be completed and the order of integration inverted to yield

$$F_p = \frac{K_z}{\sqrt{\pi}} \int_0^{\infty} d\tau'\, e^{i\tau'(n\Omega+\omega) - \frac{K_z^2KT\tau'^2}{2m}} \int_{-\infty}^{\infty} e^{-\frac{m}{2KT}\left(V_z + i\frac{K_zKT}{m}\tau'\right)^2} V_z^{\,p}\, dV_z$$

In this form it is easy to see that $F_p$ is finite and converges rapidly for large values of $\tau'$ and $V_z$. The integration of 2.47-2.49 can be carried out to yield

$$\langle f_{31} V_z \rangle = \rho e^{-\lambda} \sum_{n=-\infty}^{\infty} \phi_{11} F_1 + \phi_{12} F_2 + \phi_{13} F_3 \tag{2.50}$$

$$\langle f_{31} v_x \rangle = \rho e^{-\lambda} \sum_{n=-\infty}^{\infty} \phi_{21} F_0 + \phi_{22} F_1 + \phi_{23} F_2 \tag{2.51}$$

$$\langle f_{31} V_y \rangle = \rho e^{-\lambda} \sum_{n=-\infty}^{\infty} \phi_{31} F_0 + \phi_{32} F_1 + \phi_{33} F_2 \tag{2.52}$$

where
$$\rho = \frac{Z^2 e^2 E_{1x} E_{2x}^{*}}{K_z m KT}\, e^{i(\underline{k}\cdot\underline{r} - \omega t)} \left(\frac{m}{2KT}\right)^{1/2}$$

In the regime under consideration in this thesis $\lambda \gg 1$.

By using this assumption, the Bessel Function of imaginary argument can be expanded as[39]

$$I_n(\lambda) \rightarrow e^{\lambda} / \sqrt{2\pi\lambda}$$

Through the use of this expansion

$$\gamma_{11} = \gamma_{22} = \gamma_{31} = \gamma_{32} \rightarrow 0$$

$$\gamma_{12} = \frac{4KT e^{\lambda}}{m\alpha^2\sqrt{2\pi\lambda}} \qquad \gamma_{21} = -\frac{2KT e^{\lambda}}{m\alpha^2\sqrt{2\pi\lambda}}$$

In this order of approximation

$$\langle f_{31} V_z \rangle = \frac{\rho}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \frac{-i}{(n\Omega+\omega_1)} \left\{ \left(1 + \frac{KT K_z K_{z_1}}{m\omega_2(n\Omega+\omega_1)}\right) F_1 \right. \tag{2.53}$$

$$\left. + \left(\frac{K_{z_1}}{n\Omega+\omega_1} - \frac{2K_z}{\omega_2}\left(1 - \frac{K_{z_1}^2 KT}{m(n\Omega+\omega_1)^2}\right)\right) F_2 - \frac{2K_z K_{z_1} F_3}{\omega_2(n\Omega+\omega_1)} \right\}$$

$$\langle f_{31} V_x \rangle = \frac{\rho}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \frac{-i E_{1z} KT}{m E_{1x}} \left\{ \frac{K_z F_0}{\omega_2(n\Omega+\omega_1)} \right. \tag{2.54}$$

$$\left. - \frac{1}{n\Omega+\omega_1}\left(\frac{m}{KT} - \frac{2K_{z_1}K_z}{\omega_2(n\Omega+\omega_1)}\right)\left(F_1 + \frac{F_2 K_{z_1}}{n\Omega+\omega_1}\right) \right\}$$

$$\langle f_{31} V_y \rangle = 0 \tag{2.55}$$

2.5 Evaluation of $f_{32}$

$f_{32}$ is given from 2.18 by

$$f_{32} = -\frac{Ze\epsilon}{m}\int_{-\infty}^{t}\left(\underset{\sim}{E}_1' + \frac{\underset{\sim}{v}'\times\underset{\sim}{B}_1'}{c}\right)\cdot\frac{\partial f_2^{*\prime}}{\partial\underset{\sim}{v}'}\,dt' \qquad (2.56)$$

To evaluate $f_{32}$, $f_2^*$ must first be calculated. Replacing the subscript 1 by 2 in equation 2.10, and taking the complex conjugate yields

$$f_2^{*\prime} = -\frac{Ze\epsilon}{m}\int_{-\infty}^{t'}\left(\underset{\sim}{E}_2^{*\prime\prime} + \frac{\underset{\sim}{v}''\times\underset{\sim}{B}_2^{*\prime\prime}}{c}\right)\cdot\frac{\partial f_0''}{\partial\underset{\sim}{v}''}\,dt'' \qquad (2.57)$$

Using Maxwell's equation,

$$\underset{\sim}{K}_2\times\underset{\sim}{E}_2^* = \frac{\omega_2}{c}\underset{\sim}{B}_2^*$$

equation 2.57 can be written in terms of $E_2^*$ alone. Changing from Lagrangian to Eulerian co-ordinates then yeilds

$$f_2^{*\prime} = \frac{Ze\epsilon}{m}E_{2x}^*\,e^{i(K_2\cdot r' - \omega_2 t')}\int_0^{\infty} e^{i(v_z'K_2-\omega_2)\tau} f_0(v')(v_x'\cos\Omega\tau - \epsilon v_y'\sin\Omega\tau)\,d\tau$$

Using Maxwell's equation to eliminate $\underset{\sim}{B}_1$ in terms of $\underset{\sim}{E}_1$ in and expressing double primed quantities in terms of primed quantities yields

$$f_{32} = \frac{Z^2e^2E_{1x}E_{2x}^*}{mKT}\int_0^{\infty}\int_{-\infty}^{t} e^{i(\underline{K}\cdot r' - \omega t') + i(K_2v_z - \omega_2)\tau}\left\{\frac{1}{2}\left((1+\epsilon)e^{i\Omega\tau} + (1-\epsilon)e^{-i\Omega\tau}\right)\left(\left(\frac{v_x'+iv_y'}{2}\right)\right.\right.$$

$$\left(\left(\frac{v_x'm}{KT} + \frac{E_{1z}v_z'm}{E_{1x}KT}\right) - \tau\left(\frac{iK_{z1}}{E_{1x}}\left(E_{1z} + \frac{v_x'}{\omega_1}(E_{1x}K_{1z} - E_{1z}K_{1x})\right)\right)\right) \qquad (2.58)$$

$$-\left(1 + \frac{v_z'}{E_{1x}\omega_1}(E_{1z}K_{1x} - K_{1z}E_{1x})\right)\Big) + \frac{1}{2}\left((1+\epsilon)e^{-i\Omega\tau} + (1-\epsilon)e^{+i\Omega\tau}\right)$$

$$\left(\left(\frac{v_x'-iv_y'}{2}\right)\left(\left(\frac{v_x'm}{KT} + v_z'\frac{mE_{1z}}{E_{1x}KT}\right) - \tau\left(\frac{iK_{z1}}{E_{1x}}\left(E_{1z} + \frac{v_x'}{\omega}(E_{1x}K_{1z}\right.\right.\right.\right.$$

$$- E_{1z} K_{1x} ))))) - \left( 1 + \frac{V_z'}{E_{1x}\omega_1} ( E_{1z}K_{1x} - E_{1x}K_{1z} ) ) \right) \Big\} d\tau dt'$$

$$\text{\&} \quad \tau = t' - t''$$

It is now possible to carry out the integration with respect to $\tau$ . The integrals involved are of the form

$$I_2 = \int_0^\infty e^{i(K_2 V_z' - \omega_2 + \Omega)\tau} (\quad) d\tau$$

$$I_1 = \int_0^\infty e^{i(K_2 V_z' - \omega_2 - \Omega)\tau} (\quad) d\tau$$

$$I_3 = \int_0^\infty e^{i(K_2 V_z' - \omega_2 + \Omega)\tau} \tau (\quad) d\tau$$

$$I_4 = \int_0^\infty e^{i(K_2 V_z' - \omega_2 - \Omega)\tau} \tau (\quad) d\tau$$

where $(\ )$ is independent of $\tau$ . If $\omega_2$ is assumed to have a small negative imaginary component, then the integrals can be evaluated to give

$$I_1 = \frac{-i(\quad)}{(\omega_2 - K_2 V_z' + \Omega)} \qquad I_2 = \frac{-i(\quad)}{(\omega_2 - K_2 V_z' - \Omega)}$$

$$I_3 = \frac{-(\quad)}{(\omega_2 - K_2 V_z' + \Omega)^2} \qquad I_4 = \frac{-(\quad)}{(\omega_2 - K_2 V_z' - \Omega)^2}$$

We can assymptotically expand the integral to get

$$I_1 = -i\frac{(\ \ )}{(\omega_2+\Omega)}\left(1+\frac{K_2 V_z'}{(\omega_2+\Omega)}\right) \qquad I_2 = -\frac{i(\ \ )}{(\omega_2-\Omega)}\left(1+\frac{K_2 V_z'}{\omega_2-\Omega}\right)$$

$$I_3 = -\frac{(\ \ )}{(\omega_2+\Omega)^2}\left(1+\frac{2K_2 V_z'}{(\omega_2+\Omega)}\right) \qquad I_4 = -\frac{(\ \ )}{(\omega_2-\Omega)^2}\left(1+\frac{2K_2 V_z'}{(\omega_2-\Omega)}\right) \tag{2.59}$$

Using the results in 2.59 and unravelling the Lagrangian coordinates in 2.58, and finally collecting terms of the various powers of $V_x \pm V_y$ , $f_{32}$ becomes

$$f_{32} = \frac{Z^2 e^2 E_{1x} E_{2x}^* e^{i(\underline{K}\cdot\underline{r}-\omega t)}}{mKT} \int_0^\infty e^{i(a_1 V_x + b_1 V_y + c_1 V_z) + i\omega\tau'} f_0(v) \tag{2.60}$$

$$\left\{ B_{21} + B_{22}V_x + B_{23}V_y + B_{24}V_x^2 + B_{25}V_y^2 + B_{26}V_xV_y \right\} d\tau'$$

$$B_{21} = -\frac{1}{2}(I_1+I_2)\left(1+\frac{V_z}{E_{1x}\omega_1}\left(E_{1z}K_{1x} - E_{1x}K_{1z}\right)\right)$$

$$B_{22} = \delta_{21} e^{i\Omega\tau'} + \delta_{22} e^{-i\Omega\tau'} \qquad B_{23} = i\epsilon\left(\delta_{21} e^{i\Omega\tau'} - \delta_{22} e^{-i\Omega\tau'}\right)$$

$$B_{24} = \delta_{11} + \delta_{12} e^{i2\Omega\tau'} + \delta_{13} e^{-i2\Omega\tau'} \qquad B_{25} = \delta_{11} - \delta_{12} e^{i2\Omega\tau'} - \delta_{13} e^{-i2\Omega\tau'}$$

$$B_{26} = 2i\epsilon\left(\delta_{12} e^{i2\Omega\tau'} - \delta_{13} e^{-i2\Omega\tau'}\right)$$

$$a_1 = -\frac{K_{x_1}}{\Omega}\sin\Omega\tau' \qquad b_1 = \frac{K_{x_1}}{\Omega}\left[1-\cos\Omega\tau'\right]$$

$$\tau' = t - t'$$

where

$$\delta_{11} = \frac{m}{4KT}(I_1+I_2) - \frac{iK_2}{4E_{1x}\omega_1}(E_{1x}K_{1z} - E_{1z}K_{1x})(I_3+I_4)$$

$$\delta_{12} = \frac{m}{4KT}I_1 - \frac{iK_2}{4E_{1x}\omega_1}(E_{1x}K_{1z} - E_{1z}K_{1x})(I_3)$$

$$\delta_{13} = \frac{m}{4KT}I_2 - \frac{iK_2}{4E_{1x}\omega_1}(E_{1x}K_{1z} - E_{1z}K_{1x})I_4$$

$$\delta_{21} = \frac{V_z' E_{1z} m I_1}{2E_{1x}KT} - i\frac{K_2 E_{1z} I_3}{2E_{1x}}$$

$$\delta_{22} = \frac{V_z E_{1z} m I_2}{2E_{1x}KT} - i\frac{K_2 E_{1z} I_4}{2E_{1x}}$$

## 2.6 Velocity Moments of $f_{32}$

The results from equations 2.34 and 2.41 may be used to express the velocity moments of $f_{32}$ over $V_x, V_y$ as follows

$$\langle f_{32}\rangle_\perp = \frac{z^2e^2E_{1x}E_{2x}^*}{mKT}e^{i(\underline{k}\cdot\underline{r}-\omega t)}f_0(V_z)\int_0^\infty e^{i(c_1V_z+\omega\tau')-\lambda(1-\cos\Omega\tau')}\Big\{\beta_{21} \tag{2.61}$$

$$+\beta_{22}\,i\frac{a_1KT}{m}+\beta_{23}\,i\frac{b_1KT}{m}+\beta_{24}\left(\frac{KT}{m}-\frac{a_1^2K^2T^2}{m^2}\right)$$

$$+\beta_{25}\left(\frac{KT}{m}-\frac{b_1^2K^2T^2}{m^2}\right)-\beta_{26}a_1b_1\frac{K^2T^2}{m^2}\Big\}d\tau'$$

$$\langle f_{32}V_x\rangle_\perp = \frac{z^2e^2E_{1x}E_{2x}^*}{mKT}e^{i(\underline{k}\cdot\underline{r}-\omega t)}f_0(V_z)\int_0^\infty e^{i(c_1V_z+\omega\tau')-\lambda(1-\cos\Omega\tau')}\Big\{\beta_{21}\,i\frac{a_1KT}{m} \tag{2.62}$$

$$+\beta_{22}\left(\frac{KT}{m}-\frac{a_1^2K^2T^2}{m^2}\right)-\beta_{23}\frac{a_1b_1K^2T^2}{m^2}+\beta_{24}\,i\frac{a_1KT}{m}\left(\frac{3KT}{m}-\frac{a_1^2K^2T^2}{m^2}\right)$$

$$+ \beta_{25}\, i\frac{a_1 kT}{m}\left(\frac{kT}{m} - \frac{b_1^2 k^2T^2}{m^2}\right) + \beta_{26}\, i\frac{b_1 kT}{m}\left(\frac{kT}{m} - \frac{a_1^2 k^2T^2}{m^2}\right)\Bigg\} d\tau'$$

$$\langle f_{32} v_y\rangle_\perp = \frac{z^2e^2E_{1x}E_{2x}^* e^{i(k\cdot r-\omega t)}}{mkT} f_0(v_z) \int_0^\infty e^{i(c_1v_z+\omega\tau') - \lambda(1-\cos\Omega\tau')} \Bigg\{ \beta_{21}\, i\frac{b_1 kT}{m} \tag{2.63}$$

$$- \beta_{22} b_1 a_1 \frac{k^2T^2}{m^2} + \beta_{23}\left(\frac{kT}{m} - \frac{b_1^2k^2T^2}{m^2}\right) + \beta_{24}\, i b_1\frac{kT}{m}\left(\frac{kT}{m} - a_1^2\frac{k^2T^2}{m^2}\right)$$

$$+ \beta_{25}\, i b_1\frac{kT}{m}\left(\frac{3kT}{m} - b_1^2\frac{k^2T^2}{m^2}\right) + \beta_{26}\, i a_1\frac{kT}{m}\left(\frac{kT}{m} - b_1^2\frac{k^2T^2}{m^2}\right)\Bigg\} d\tau'$$

The term $e^{\lambda\cos\Omega\tau'}$ can again be expanded by using

$$e^{\lambda\cos\Omega\tau'} = \sum_{n=-\infty}^{\infty} I_n(\lambda) e^{in\Omega\tau'}$$

Using equations 2.41 and rearranging the infinite sums of

Bessel functions, we can write 2.61-2.63 as

$$\langle f_{32}\rangle_\perp = \frac{z^2e^2E_{1x}E_{2x}^* e^{i(k\cdot r-\omega t)-\lambda}}{mkT} \sum_{n=-\infty}^{\infty} \int_0^\infty f_0(v_z) \tag{2.64}$$

$$e^{i(\omega-k_zv_z+n\Omega)\tau'} \Bigg\{ \delta_{11}\left(\frac{2kT}{m} I_n + \alpha^2\left(4(I_{n-1}+I_{n+1}) - 8I_n\right)\right)$$

$$+ \delta_{12}\alpha^2\left(4(I_{n-2}+I_n) - 8I_{n-1}\right) + \delta_{13}\alpha^2\left(4(I_{n+2}+I_n) - 8I_{n+1}\right)$$

$$- 2\delta_{21}\alpha\left(I_{n-1}-I_n\right) - 2\delta_{22}\alpha\left(I_n - I_{n+1}\right) + \beta_{21} I_n\Bigg\} d\tau'$$

$$\langle f_{32} v_x\rangle_\perp = \frac{z^2e^2E_{1x}E_{2x}^* e^{i(k\cdot r-\omega t)-\lambda}}{mkT} \sum_{n=-\infty}^{\infty} \int_0^\infty f_0(v_z)$$

$$e^{i(\omega-k_zv_z+n\Omega)\tau'} \Bigg\{ \delta_{11}\left(-\frac{4kT\alpha}{m}\left(I_{n-1}-I_{n+1}\right)\right.$$

$$+ \alpha^3\left( 8(I_{n-1}-I_{n+1}) - 4(I_{n-2}-I_{n+2})\right)\Big) + \delta_{12}\Big(-\frac{4\alpha KT}{m}$$

$$(I_{n-2}-I_{n-1}) - \alpha^3\left( 4(I_{n-3}-I_{n+1}) + 8(I_n - I_{n-2})\right)\Big)$$

(2.65)

$$+ \delta_{13}\Big(-\frac{4\alpha KT}{m}(I_{n+1}-I_{n+2}) - \alpha^3\left( 4(I_{n-1}-I_{n+3}) + 8(I_{n+2}-I_n)\right)\Big)$$

$$+ \delta_{21}\Big( \frac{KT}{m} I_{n-1} + 2\alpha^2(I_{n+1}-I_n-I_{n-1}+I_{n-2})\Big) - \alpha\beta_{21}(I_{n-1}-I_{n+1})$$

$$+ \delta_{22}\Big( \frac{KT}{m} I_{n+1} + 2\alpha^2(I_{n-1}-I_n-I_{n+1}+I_{n+2})\Big)\Big\} d\tau'$$

$$\langle f_{32} V_y \rangle_\perp = \frac{z^2 e^2 E_{1x} E_{2x}^* \, e^{i(k\cdot r - \omega t) - \lambda}}{mKT} \sum_{n=-\infty}^{\infty} \int_0^\infty f_0(v_z)$$

$$\Big\{ \delta_{11}\Big( \frac{4i\epsilon\alpha KT}{m}(2I_n - I_{n-1} - I_{n+1}) + i\epsilon\alpha^3\big( 16(I_{n-1}+I_{n+1})$$

$$- 24 I_n - 4(I_{n-2}+I_{n+2})\big)\Big) + \delta_{12}\Big(-\frac{4i\epsilon\alpha KT}{m}(I_{n-2}-I_{n-1})$$

$$+ i\epsilon\alpha^3\big( 16(I_{n-2}+I_n) - 4(I_{n+1}+I_{n-3}) - 24 I_{n-1}\big)\Big)$$

(2.66)

$$+ \delta_{13}\Big(-\frac{4i\epsilon\alpha KT}{m}(I_{n+2}-I_{n+1}) + i\epsilon\alpha^3\big( 16(I_{n+2}+I_n)$$

$$- 4(I_{n-1}+I_{n+3}) - 24 I_{n+1}\big)\Big) + \delta_{21}\Big( i\epsilon\frac{KT}{m} I_{n-1} - i\epsilon\alpha^2\big( 6(I_{n-1}$$

$$- I_n) + 2(I_{n+1}-I_{n-2})\big)\Big) + \delta_{22}\Big( i\epsilon\frac{KT}{m} I_{n+1} - i\epsilon\alpha^2\big(-6(I_{n+1}$$

$$- I_n + 2(I_{n+2}-I_{n-1})\big)\Big) + i\epsilon\alpha\beta_{21}(2I_n - I_{n+1} - I_{n-1})\Big\} d\tau'$$

To complete the calculation of the moments over $f_{32}$, in the regime of interest in this study, the integations over $V_z$ and $\tau'$ are performed and $I_n(\lambda)$ is replaced by its large argument asymptotic expansion( $I_n(\lambda) \rightarrow e^{\lambda}/\sqrt{2\pi\lambda}$). The integrals over $V_z$ and $\tau'$ are of the standard form $F_0, F_1 \ldots$ .

$$\langle f_{32} V_z \rangle = \frac{\rho}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \theta_{11} F_1 + \theta_{12} F_2 + \theta_{13} F_3 \tag{2.67}$$

$$\langle f_{32} V_x \rangle = \frac{\rho}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \theta_{21} F_0 + \theta_{22} F_1 + \theta_{23} F_2 \tag{2.68}$$

$$\langle f_{32} V_y \rangle = \epsilon \langle f_{32} V_x \rangle \tag{2.69}$$

where

$$\theta_{11} = \frac{i K_2 KT (E_{1x} K_{1z} - E_{1z} K_{1x})}{2 E_{1x} \omega_1 \omega_2^2} \left\{ \left( \frac{\omega_2}{\omega_2 + \Omega} \right)^2 + \left( \frac{\omega_2}{\omega_2 - \Omega} \right)^2 \right\}$$

$$\theta_{12} = \frac{i (E_{1x} K_{1z} - K_{1x} E_{1z})}{2 E_{1x} \omega_1 \omega_2} \left\{ \frac{KT K_2^2}{m \omega_2^2} \left( \left( \frac{\omega_2}{\omega_2 - \Omega} \right)^2 + \left( \frac{\omega_2}{\omega_2 + \Omega} \right)^2 \right) - \left( \frac{\omega_2}{\omega_2 - \Omega} + \frac{\omega_2}{\omega_2 + \Omega} \right) \right\}$$

$$\theta_{13} = \frac{-i K_2 (E_{1x} K_{1z} - E_{1z} K_{1x})}{2 \omega_2^2 \omega_1 E_{1x}} \left\{ \left( \frac{\omega_2}{\omega_2 - \Omega} \right)^2 + \left( \frac{\omega_2}{\omega_2 + \Omega} \right)^2 \right\}$$

$$\theta_{21} = \frac{i E_{1z} KT}{2 E_{1x} \omega_2 C m} \left( \left( \frac{\omega_2}{\omega_2 - \Omega} \right)^2 + \left( \frac{\omega_2}{\omega_2 - \Omega} \right)^2 \right)$$

$$\theta_{22} = \frac{-iE_{1z}}{2E_{1x}\omega_2}\left\{\frac{\omega_2}{\omega_2-\Omega} + \frac{\omega_2}{\omega_2+\Omega} + \frac{2KT}{mc^2}\left(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^3 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^3\right)\right\}$$

$$\theta_{23} = \frac{-iE_{1z}}{2E_{1x}c\,\omega_2}\left(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\right)$$

## 2.7 Evaluation of Driving Currents

Equations 2.53-2.55 and equations 2.67-2.69 can be now used to express the total second-order driving current as

$$\underset{\sim}{J}_{3d} = \sum_k n_0 e\left(\langle \underset{\sim}{v}\, f^+_{3k}\rangle - \langle \underset{\sim}{v}\, f^-_{3k}\rangle\right) \tag{2.70}$$

where $K=1,2$.

Equation 2.73 can be expressed as

$$\underset{\sim}{J}_{3d} = \frac{\omega D^-}{4\pi}\sum_{n=-\infty}^{\infty}\left(\left(\frac{m_-}{m_+}\right)^{3/2}\pi^+_{in} - \pi^-_{in}\right)\hat{\epsilon}_i \tag{2.71}$$

where $i=1,2,3$ refers to the $x, y, z$ components of the current.

$$\pi^{\pm}_{in} = \frac{iE_{1z}}{E_{1x}\omega_2}\left(\frac{KT}{m}\right)^{1/2}\left\{\left(\frac{1}{2c}\left(\frac{KT}{m}\right)^{1/2}\left(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\right)\right.\right.$$

$$\left.-\frac{K_z}{(n\Omega+\omega_1)}\left(\frac{KT}{m}\right)^{1/2}\right)F_0 + \left(F_0\left(\frac{\omega+n\Omega}{K_z}\right)\left(\frac{m}{KT}\right)^{1/2} - i\sqrt{2}\right) \tag{2.72}$$

$$\left(\frac{\omega_2}{\omega_1+n\Omega}\left(1 - \frac{2K_{z_1}KT}{c(n\Omega+\omega_1)m}\right)\left(1 + \frac{K_{z_1}(\omega+n\Omega)}{(\omega_1+n\Omega)K_z}\right) - \frac{1}{2}\left(\frac{\omega_2}{\omega_2+\Omega} + \frac{\omega_2}{\omega_2-\Omega}\right)\right.$$

$$\left.\left.-\frac{\omega+n\Omega}{2K_zc}\left(\left(\frac{\omega_2}{\omega_2+\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2-\Omega}\right)^2\right) + \frac{KT}{mc^2}\left(\left(\frac{\omega_2}{\omega_2+\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2-\Omega}\right)^2\right)\right)\right\}$$

$$\pi_{2n}^{\pm} = \frac{\epsilon i E_{1z}}{\omega_2 E_{1x}}\left(\frac{kT}{m}\right)^{1/2}\Bigg\{ \frac{F_0}{2c}\left(\frac{kT}{m}\right)^{1/2}\Bigg(\left(\frac{\omega_2}{\omega_2+\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2-\Omega}\right)^2\Bigg)$$

$$+\Bigg(F_0\left(\frac{\omega+n\Omega}{k_z}\right)\left(\frac{m}{kT}\right)^{1/2} - i\sqrt{2}\Bigg)\Bigg( \frac{kT}{mc^2}\Bigg(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\Bigg)$$

$$-\frac{1}{2}\Bigg(\frac{\omega_2}{\omega_2+\Omega} + \frac{\omega_2}{\omega_2-\Omega}\Bigg) - \frac{(\omega+n\Omega)}{2ck_z}\Bigg(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\Bigg)\Bigg)\Bigg\} \qquad (2.73)$$

$$\pi_{3n}^{\pm} = \frac{i}{\omega_2}\left(\frac{kT}{m}\right)^{1/2}\Bigg\{\Bigg(F_0\left(\frac{\omega+n\Omega}{k_z}\right)\left(\frac{m}{kT}\right)^{1/2} - i\sqrt{2}\Bigg)\Bigg(\frac{(E_{1x}k_{1z}-E_{1z}k_{1x})}{2E_{1x}\omega_1 c}\frac{kT}{m}$$

$$\Bigg(\Bigg(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\Bigg) + \left(\frac{\omega+n\Omega}{k_z c}\right)\Bigg(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^3 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^3\Bigg)$$

$$-\frac{cm}{kT}\left(\frac{\omega+n\Omega}{k_z}\right)\Bigg(\frac{\omega_2}{\omega_2-\Omega} + \frac{\omega_2}{\omega_2+\Omega}\Bigg) - \frac{m}{kT}\left(\frac{\omega+n\Omega}{k_z}\right)^2\Bigg(\left(\frac{\omega_2}{\omega_2+\Omega}\right)^2$$

$$+\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2\Bigg)\Bigg)\Bigg) - \frac{\omega_2}{(\omega_1+n\Omega)}\Bigg(1 + \frac{kTk_{z1}}{mc(n\Omega+\omega_1)} + \frac{k_{z1}(\omega+n\Omega)}{(\omega_1+n\Omega)k_z} \qquad (2.74)$$

$$-\frac{2(\omega+n\Omega)}{ck_z}\Bigg(1 - \frac{k_{z1}kT}{m(n\Omega+\omega_1)^2}\Bigg) - \frac{2k_{z1}}{c(n\Omega+\omega_1)}\left(\frac{\omega+n\Omega}{k_z}\right)^2\Bigg)\Bigg)$$

$$-i\sqrt{2}\Bigg(\frac{(E_{1x}k_{1z}-E_{1z}k_{1x})}{2E_{1x}\omega_1 c}\frac{kT}{m}\Bigg(\left(\frac{\omega_2}{\omega_2-\Omega}\right)^2 + \left(\frac{\omega_2}{\omega_2+\Omega}\right)^2\Bigg)$$

$$-\frac{2k_{z1}\omega_2 kT}{mc(n\Omega+\omega_1)^2}\Bigg)\Bigg\}$$

and

$$D^{-} = \frac{\omega_{p-}^{2} E_{1x} E_{2x}^{*} e}{\omega\, KT\, K_z \sqrt{2\pi\lambda}} \left(\frac{m_-}{KT}\right)^{1/2} e^{i(\underline{k}\cdot\underline{r}-\omega t)} \tag{2.75}$$

2.8 Velocity Moments of $f_{33}$

From the equation 2.5, $f_{33}$ is found to be given by

$$f_{33} = -\frac{Zee}{m}\int_{-\infty}^{t}\left(\underline{E}_3^{*} + \frac{\underline{v}' \times \underline{B}_3}{c}\right)\cdot \frac{\partial f_0'}{\partial \underline{v}'}\, dt'$$

The integration of this equation is done in Stix,[40] and the results that are applicable to this case are

$$\langle f_{33} v_x \rangle = \frac{Zee}{K_z m}\left(\frac{m}{KT}\right)^{1/2} \frac{E_{3x}\, e^{i(\underline{k}\cdot\underline{r}-\omega t)}}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} F_0 \tag{2.76}$$

$$\langle f_{33} v_y \rangle = \langle f_{33} v_x \rangle \frac{E_{3y}}{E_{3x}} \tag{2.77}$$

$$\langle f_{33} v_x \rangle = \frac{Zee}{K_z m}\left(\frac{m}{KT}\right)^{1/2} \frac{E_{3z}\, e^{i(\underline{k}\cdot\underline{r}-\omega t)}}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \left(\frac{m}{KT}\right) F_2 \tag{2.78}$$

[Note; the results given by equation 2.76-2.78 are obtained from Stix with a Maxwellian velocity distribution assumed in the z-direction and approximating $I_n(\lambda)$ by its asymptotic value $\frac{e^{\lambda}}{\sqrt{2\pi\lambda}}$.]

Following the analysis in Stix, the expectation value of the velocity over the second-order induced distribution function can be expressed as

$$\langle \underset{\rightarrow}{v}\, f_{33}^{\pm} \rangle = \frac{c}{B_o} \underset{\Rightarrow}{M}_3^{\pm} \cdot \underset{\rightarrow}{E}_3 \qquad (2.79)$$

where the various components of the mobility tensor are given by

$$M_{3yy}^{\pm} = M_{3xx}^{\pm} = \frac{\Omega_{\pm}\epsilon}{\sqrt{2}\, k_z} \left(\frac{m}{kT}\right)^{1/2} \frac{1}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} F_o \qquad (2.80)$$

$$M_{3zz}^{\pm} = \frac{\Omega_{\pm}\epsilon}{k_z} \left(\frac{m}{2kT}\right)^{1/2} \frac{1}{\sqrt{2\pi\lambda}} \sum_{n=-\infty}^{\infty} \left(\frac{m}{kT}\right) F_2 \qquad (2.81)$$

All other components of the mobility tensor are zero.
(In the limit $I_n(\lambda) \rightarrow \dfrac{e^{\lambda}}{\sqrt{2\pi\lambda}}$ )

2.9 Calculation of Total Current

Using 2.79, the second-order induced current can be expressed as

$$J_{in} = \frac{\omega_{p+}^2}{4\pi\Omega_+} \underset{\Rightarrow}{M}_3^{+} \cdot \underset{\rightarrow}{E}_3 - \frac{\omega_{p-}^2}{4\pi\Omega_-} \underset{\Rightarrow}{M}_3 \cdot \underset{\rightarrow}{E}_3 \qquad (2.82)$$

The total second-order current for each species is then given by

$$J_3^{\pm} = \frac{\omega_{p\pm}^2}{4\pi\Omega_{\pm}} \underset{\Rightarrow}{M}_3^{\pm} \cdot M + \underset{\rightarrow}{J}_{3d}^{\pm} \qquad (2.83)$$

and the total current is given by

$$\underset{\to}{J}_3 = \left( \frac{\omega_{p+}^2}{4\pi\Omega_+} \underset{\to}{M}_3^+ - \frac{\omega_{p-}^2}{4\pi\Omega_-} \underset{\to}{M}_3^- \right) \cdot \underset{\to}{E}_3 + \underset{\to}{J}_{3d}^+ + \underset{\to}{J}_{3d}^- \tag{2.84}$$

## 2.10 Solution of Second-Order Fields

Once the second-order currents have been calculated, Maxwell's equation can be solved for the second-order field quantities.

$$\nabla \times \underset{\to}{E}_3 = -\frac{1}{c}\frac{\partial \underset{\to}{B}_3}{\partial t} \quad \& \quad \nabla \times \underset{\to}{B}_3 = \frac{1}{c}\frac{\partial \underset{\to}{E}_3}{\partial t} + \frac{4\pi}{c}\underset{\to}{J}_3 \tag{2.85}$$

Fourier analysing 2.85 and using the refractive index

$$\underset{\to}{n} = \frac{\underset{\to}{k}c}{\omega}$$

yields

$$(\underset{\to}{n}\cdot\underset{\to}{E}_3)\underset{\to}{n} - \underset{\to}{E}_3 n^2 + \left[1 + \frac{i}{\omega}\left( \frac{\omega_{p+}^2}{\Omega_+}\underset{\to}{M}_3^+ \cdot - \frac{\omega_{p-}^2}{\Omega_-}\underset{\to}{M}_3^- \cdot \right)\right]\underset{\to}{E}_3 = \frac{4\pi}{i\omega}\underset{\to}{J}_D$$

In tensorial form, this equation becomes

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{bmatrix} \begin{bmatrix} E_{3x} \\ E_{3y} \\ E_{3z} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \tag{2.86}$$

where

$$A_{11} = 1 - n_z^2 + \frac{i}{\omega}\left( \frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^+ - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^- \right)$$

$$A_{22} = 1 - n^2 + \frac{i}{\omega}\left( \frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^+ - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^- \right)$$

$$A_{33} = 1 - n_x^2 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3zz}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3zz}^{-}\right)$$

$$A_{31} = A_{13} = n_x n_z \tag{2.87}$$

and

$$R_1 = \frac{4\pi}{i\omega} J_{dx} \qquad R_2 = \frac{4\pi}{i\omega} J_{dy}$$

$$R_3 = \frac{4\pi}{i\omega} J_{dz} \tag{2.88}$$

Solving for

$$\begin{bmatrix} E_{3x} \\ E_{3y} \\ E_{3z} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} C_{11} & 0 & C_{13} \\ 0 & C_{22} & 0 \\ C_{31} & 0 & C_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \tag{2.89}$$

$$\Delta = n_x^2\left(n_z^2 - 1 - \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^{-}\right)\right)$$

$$+ n_z^2\left(n_x^2 - 1 - \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3zz}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3zz}^{-}\right)\right)$$

$$+ \left[1 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^{-}\right)\right]\left[1 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3zz}^{+}\right.\right.$$

$$\left.\left. - \frac{\omega_{p-}^2}{\Omega_-} M_{3zz}^{-}\right)\right]$$

$$C_{11} = 1 - n_z^2 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3zz}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3zz}^{-}\right) \tag{2.90}$$

$$C_{13} = C_{31} = n_x n_z$$

$$C_{22} = \frac{C_{11}\left(1 - n_x^2 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3zz}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3zz}^{-}\right)\right) - n_x^2 n_z^2}{1 - n^2 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^{-}\right)}$$

$$C_{33} = 1 - n_z^2 + \frac{i}{\omega}\left(\frac{\omega_{p+}^2}{\Omega_+} M_{3xx}^{+} - \frac{\omega_{p-}^2}{\Omega_-} M_{3xx}^{-}\right)$$

By using 2.90 in 2.89, the electric fields may be obtained in terms of the driving fields.

## 2.11 Calculation of Energy Absorption

The power absorbed by the second-order induced currents can be calculated by using equation 2.89 for the second-order electric field and by using equation 2.83 for second-order induced currents. The power absorbed by the ions is

$$W_3^{ion} = \frac{\omega_{p+}^2}{4\pi\Omega_+}\left(\left(|E_{3x}|^2 + |E_{3y}|^2\right) \mathrm{Re}\, M_{3xx}^{+*} + |E_{3z}|^2 \mathrm{Re}\, M_{3zz}^{+*}\right) \tag{2.91}$$

The power absorbed by the electrons is given by

$$W_3^{electron} = \frac{\omega_{p-}^2}{4\pi\Omega_-}\left(\left(|E_{3x}|^2 + |E_{3y}|^2\right) \mathrm{Re}\, M_{3xx}^{-*} + |E_{3z}|^2 \mathrm{Re}\, M_{3zz}^{-*}\right) \tag{2.92}$$

and the total power absorbed is

$$W_3 = \left(|E_{3x}|^2 + |E_{3y}|^2\right) \mathrm{Re}\left\{ \frac{\omega_{p+}^2}{4\pi\Omega_+} M_{3xx}^{+*} - \frac{\omega_{p-}^2}{4\pi\Omega_-} M_{3xx}^{-*} \right\}$$

$$+ |E_{3z}|^2 \, \mathrm{Re}\left\{ \frac{\omega_{p+}^2}{4\pi\Omega_+} M_{3zz}^{+*} - \frac{\omega_{p-}^2}{4\pi\Omega_-} M_{3zz}^{-} \right\} \qquad (2.93)$$

# CHAPTER 3 DISCUSSION OF RESULTS

This chapter includes, as part of a discussion of the feasibility of ion cyclotron heating:

1. A comparison of collisional versus collisionless energy absorption of the difference frequency wave.
2. A comparison between first and second-order energy absorption by the ions.
3. A complete analysis of the dependence of absorbed power on number density, temperature and fluctuations in magnetic field, incident angle, and difference frequency.
4. Discussion of results and suggestions for further research.

## 3.1 Comparison Between Collisionless and Collisional Absorption

The Stix formula[39] for modifying the (dimensionless) mobility tensors to include collional effects are

$$\Delta \underline{\underline{M}}_3^+ = \frac{\delta_{ie}}{\Omega_+} \underline{\underline{M}}_3^+ \cdot \underline{\underline{I}} \cdot \left( \underline{\underline{M}}_3^- - \underline{\underline{M}}_3^+ \right)$$
$$\Delta \underline{\underline{M}}_3^- = -\frac{\delta_{ie}}{\Omega_+} \underline{\underline{M}}_3^- \cdot \underline{\underline{I}} \cdot \left( \underline{\underline{M}}_3^+ - \underline{\underline{M}}_3^- \right) \tag{3.1}$$

The correction to the total power absorbed is then found by adding the corrections to the mobility tensors and using this new mobility tensor to evaluate the power absorbed by the ions.

Calculations of the correction to the power absorbed

by collisional effects were carried out using

$$n = 10^{14}/\text{cm}^3 \qquad B = 5000 \text{ gauss}$$

$$\theta = 15^\circ \qquad T = 10^6\ {}^\circ\text{K}$$

$$\Delta = 10^{-4} \qquad \omega_1 = 1.778 \times 10^{14} \text{ radians/sec}$$

The correction to the power absorbed was found to be less than 8%. Varying the nominal parameters (i.e. temperature, magnetic field, mixing angle and difference frequency) placed and upper bound of 10% to the ratio of collisional to collisionless ion power absorption. Due to their relative unimportance, in the temperature and density range considered in this study (for temperature between $5 \times 10^5\ {}^\circ$K to $5 \times 10^6\ {}^\circ$K and number densities between $10^{12}$/c.c. and $10^{16}$/c.c.) collisional corrections will be ignored in the remainder of this chapter.

3.2 Comparison of First and Second Order Energy Absorption From the Incident Field

A cold plasma analysis gives the power absorbed by the ion and electrons from the incident wave through collisional damping as

$$W_i^e = \frac{\omega_1 \delta_{ee} \omega_{p-}^2}{4\pi} \left\{ \frac{(\omega_1 + \Omega_+)|E_{1x} - iE_{1y}|^2}{(\omega_1 + \Omega_+)^2(\omega_1 - \Omega_-)^2 + \omega_1^2 \delta_{ee}^2} \right. \tag{3.2}$$

$$\left. + \frac{(\omega_1 - \Omega_-)|E_{1x} + E_{1y}|^2}{(\omega_1 + \Omega_-)^2(\omega_1 - \Omega_+)^2 + \omega_1^2 \delta_{ee}^2} + \frac{|E_{1z}|^2}{\omega_1^3} \right\}$$

For the case under consideration $\omega_1 \gg \Omega_- > \Omega_+$ this expression reduces to

$$W_i^e = \frac{\delta_{ee}\, \omega_{p-}^2 |E_1|^2}{4\pi \omega_1^2} \tag{3.3}$$

The collision freqency used in 3.2 can be estimated using the Spitzer formula

$$\delta_{ei} = \left(\frac{2\pi}{m_e}\right)^{1/2} n_0 e^4 \ln\Lambda \,/\, (KT_e)^{3/2} \tag{3.4}$$

Computations to compare the first-order and second-order energy absorption were carried out using an I.B.M.-360 for the following nominal values.

$$\begin{aligned} &n = 10^{14}/cm^3 && B = 5000 \text{ gauss} \\ &\Delta = 10^{-4} && \omega_1 = 1.778 \times 10^{14} \text{ (10.6}\mu \text{ line of } CO_2) \\ &\theta = 15^\circ && T = 10^6 \text{ °K} \end{aligned} \tag{3.5}$$

The resulting first-order powers, using equations 3.3-3.5, are

$$\begin{aligned} W_i^{e} &\sim 1 \times E_1^2 \\ W_i^{ion} &= 5.4 \times 10^{-4} \times E_1^2 \end{aligned} \tag{3.6}$$

The total power absorbed is

$$W_i^{total} \sim 1 \times E_1^2 \tag{3.7}$$

The power absorbed from the difference frequency harmonic [equations 2.91 and 2.92] for the parameters given in 3.5 is given by

$$W_3^{ion} = 2.7 \times 10^{-14} \times E_1^4$$

and

$$W_3^{electron} = 1.2 \times 10^{-12} \times E_1^4 \tag{3.8}$$

Comparison of first and second order ion power absorption yields

for

$$W_i^{ion} = W_3^{ion.}$$

$$5.4 \times 10^{-4} \times E_1^2 = 2.7 \times 10^{-14} \times E_1^4$$

Therefore, for $E_1 > 1.4 \times 10^5$ st./cm the second-order power absorbed by the ions is larger than the first-order power absorbed by the ions. This means that the non-linear heating takes over from linear heating when $E_1 > 1.4 \times 10^5$ statvolts/cm.

## 3.3 Dependence of Absorbed Power on Fluctuations in Plasma and Experimental Parameters

Equations 2.91 and 2.92 were used to calculate the dependence of the absorbed power on plasma and experimental parameters. The graphs shown in figures 1-4, give the results of computations carried out using an I.B.M.-360 and the following parameters

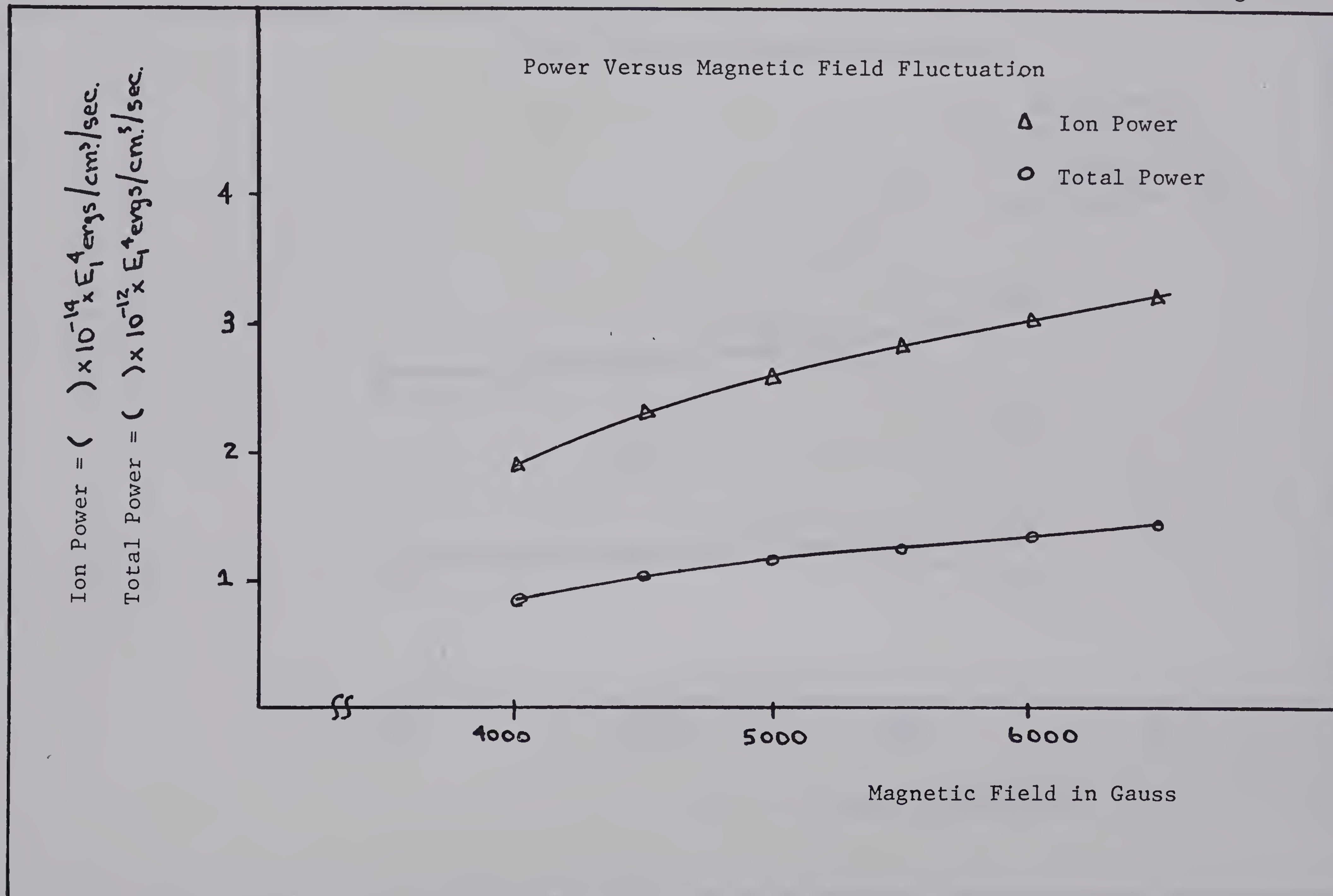
$$n = 10^{14}/\text{cm}^3$$
$$T = 10^6 \ {}^\circ K$$
$$\Delta = 10^{-4}$$
$$B = 5000 \text{ gauss}$$
$$\omega_1 = 1.778 \times 10^{14}/\text{sec.}$$
$$\theta = 15^\circ$$

In figure 5, the nominal values used were the same as above except for $\theta$, which is set equal to $20^\circ$.

The nominal value of $\theta$ in the computations was chosen to allow a larger range of temperature variation. The magnetic field and mixing angle were chosen to ensure the convergence of

$$I_n(\lambda) \text{ to } \frac{e^\lambda}{\sqrt{2\pi\lambda}} .$$

Fig. 1


Ion Power = (     ) x 10^-14 x E1^4 ergs/cm³/sec.
Total Power = (     ) x 10^-12 x E1^4 ergs/cm³/sec.
Power Versus Magnetic Field Fluctuation
Ion Power
Total Power
1
2
3
4
4000
5000
6000
Magnetic Field in Gauss

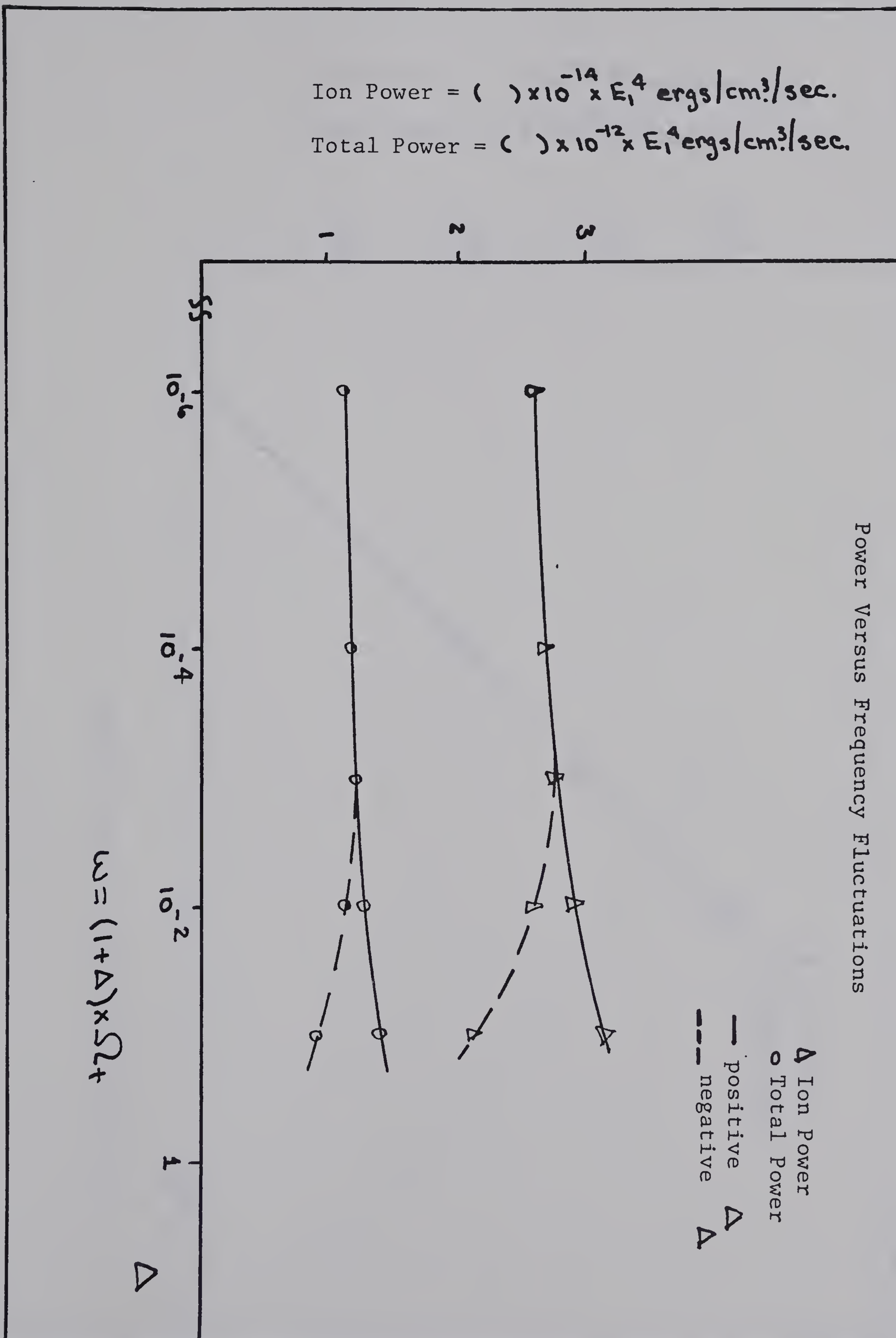

Ion Power = ( ) x $10^{-14}$ x $E_1^4$ ergs/cm$^3$/sec.
Total Power = ( ) x $10^{-12}$ x $E_1^4$ ergs/cm$^3$/sec.
1
2
3
$10^{-6}$
$10^{-4}$
$10^{-2}$
1
$\omega = (1+\Delta) \times \Omega_+$
$\Delta$
Power Versus Frequency Fluctuations
Δ Ion Power
o Total Power
— positive Δ
- - - negative Δ

Fig. 2

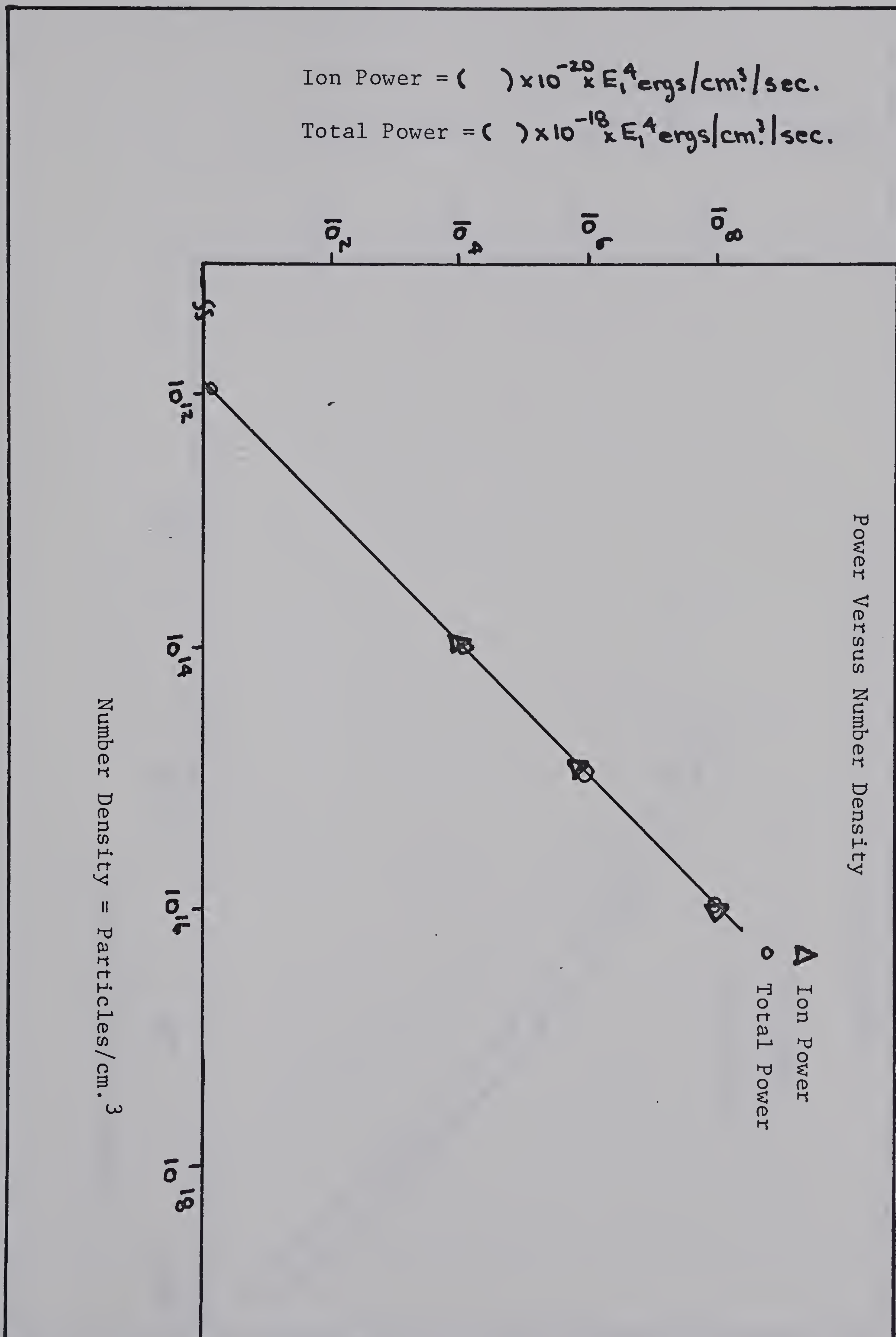

Ion Power = ( ) x 10^-20 x E1^4 ergs/cm.3/sec.
Total Power = ( ) x 10^-18 x E1^4 ergs/cm.3/sec.
Power Versus Number Density
Ion Power
Total Power
10^2
10^4
10^6
10^8
10^12
10^14
10^16
10^18
Number Density = Particles/cm.3

Fig. 3

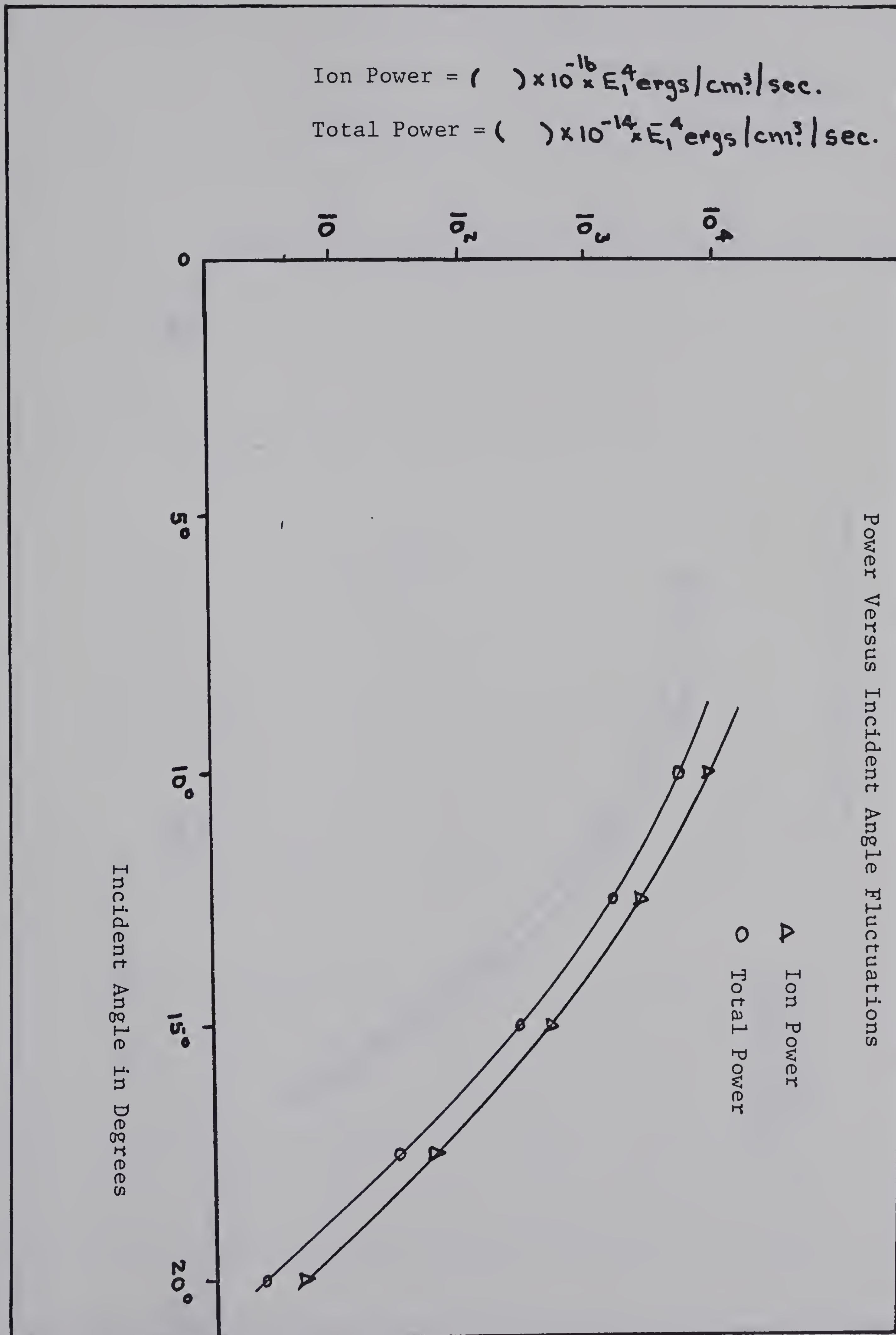

Ion Power = (   ) × $10^{-16}$ × $E_1^4$ ergs/cm³/sec.
Total Power = (   ) × $10^{-14}$ × $E_1^4$ ergs/cm³/sec.
Power Versus Incident Angle Fluctuations
Δ Ion Power
O Total Power
$10^{-1}$
$10^{-2}$
$10^{-3}$
$10^{-4}$
0
5°
10°
15°
20°
Incident Angle in Degrees

Fig. 4

Ion Power = (   ) $\times 10^{-20} \times E_1^4$ ergs/cm$^3$/sec

Total Power = (   ) $\times 10^{-18} \times E_1^4$ ergs/cm$^3$/sec

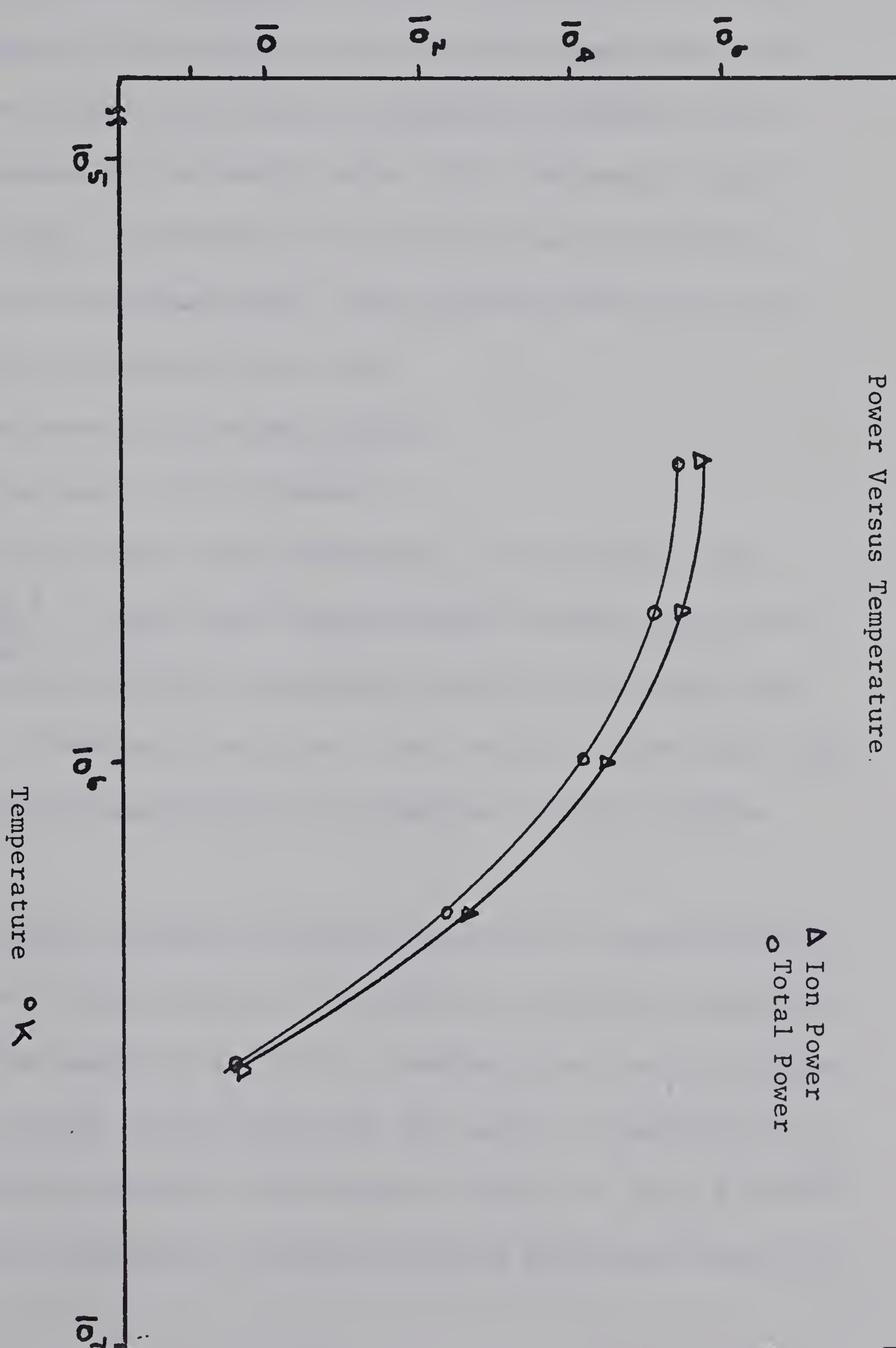



Fig. 5

The computations performed show that as $\lambda$ gets smaller, the power absorbed from the second-order fields increases. This fact is born out by observing figures 1, 4, and 5. [Note; $\lambda = \frac{K_x^2 KT}{\Omega^2 m}$ ]. In figure 1, we note the increase in absorbed power with increasing magnetic field. In figure 5, we note a large increase in the absorbed power as the temperature goes down and in figure 4, we note an increase in absorbed power with a decrease in the mixing angle. (i.e. decreasing $K_x$). Decreasing $K_x$ corresponds to increasing the perpendicular wavelength of the mixed wave. The large perpendicular wavelength can be achieved in two ways:

1. Decreasing the mixing angle
2. Decreasing the frequency

For this type of wave mixing phenomena, the absorbed power goes as $\frac{E_1^4}{\omega_1^4}$. This would indicate that the best way to increase the perpendicular wavelength would be to decrease the frequency. (Keeping in mind that $\omega_1$ must be larger than $\omega_{p-}$ to ensure good penetration of the incident fields into the plasma).

The graphs plotted in figures 1-5 show the heating scheme is relatively insensitive to fluctuations in nominal parameters. The relative insensitivity of the absorbed power to fluctuations brings the scheme under study with the realm of feasibility as a plasma heating method. For electric fields of $E_1 > 1.4 \times 10^5$ statvolts/cm. difference frequency heating takes over from col-

lisional first-order heating. Since electric fields of $1.5 \times 10^6$ statvolts/cm.[41] have been achieved in high intensity lasers, it is now possible to conduct an experiment to detect heating due to the difference frequency harmonic.

The graphs in figures 1-5, show that the power absorbed from the difference frequency wave goes mainly to the electrons. This observation leads to the conclusion that an analysis should be carried out in the regime where the difference frequency is tuned to the electron cyclotron frequency. This tuning to the electron cyclotron frequency should enhance the absorbed power by a factor at least as large as $\frac{m_i}{m_e}$. From the graphs in figures 1-5, we notice that the absorbed power increases with decreasing $\lambda$. This observation leads to the conclusion that the analysis done in this thesis should be modified such that the asymptotic expansion of the Bessel Function for small arguments could be used instead of the larger argument expansion used in this thesis.

Consideration could also be given to the mechanism by which the difference frequency wave-packet created in a semi-infinite plasma diffuses in the plasma. This study could have practical implications since in order to achieve fusion (fusion here refers to the generation of useful energy from a fusion reaction) large volumes (large compared to the spot size of a laser beam) of plasma would have to be heated to ensure that a fusion reaction could be maintained. Since the heating should be non-local, the diffusion time of the wave-packet can be an impor-

tant parameter in the use of this mechanism for plasma heating. Note; fusion reactors where the volume of the fuel (usually in the form of a solid pellet) is of the order of the spot size of a laser have real possibilities but are not included in this discussion.

The effects of difference frequency heating on velocity-space instabilities could also be considered. i.e. Does the difference wave give rise to large velocity-space anisotropies, and if so how do these anisotropies affect the macroscopic stability of the plasma? A more complete theory (for low values of $\tau$ collisional heating becomes more important) would also include a fuller consideration of collisional damping of the difference frequency wave.

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